previously. Because payment is the independent variable in the function, $\left[\left(1+\frac{r}{c}\right)^{3c-1}+\left(1+\frac{r}{c}\right)^{3c-2}+\cdots+1\right]$ is the slope. This, however, is a geometric series with first term 1, ratio $\left(1+\frac{r}{c}\right)$, and $3c$ terms, so it could be rewritten

$$\frac{1-\left(1+\frac{r}{c}\right)^{3c}}{1-\left(1+\frac{r}{c}\right)}$$

So, for any principal P and interest rate r compounded c times per year, the relationship between payment and balance after 3 years is

$$Balance3Years = \frac{1-\left(1+\frac{r}{c}\right)^{3c}}{1-\left(1+\frac{r}{c}\right)} \cdot Payment + P\left(1+\frac{r}{c}\right)^{3c}$$

You can now confirm the linear function in the preceding graph, using *Principal* = 100, *InterestRate* = 20, and *PeriodsPerYear* = 4:

$$Balance3Years = \frac{1-\left(1+\frac{0.20}{4}\right)^{3.4}}{1-\left(1+\frac{0.20}{4}\right)} \cdot Payment +$$

$$100\left(1+\frac{0.20}{4}\right)^{3.4} \approx 15.9 \cdot Payment + 180$$

Some students may recognize that the function can be further generalized for t years:

$$Balance3Years = \frac{1-\left(1+\frac{r}{c}\right)^{tc}}{1-\left(1+\frac{r}{c}\right)} \cdot Payment + P\left(1+\frac{r}{c}\right)^{tc}$$

JUPITER'S MOONS (PAGE 59)

Activity Time: 30–45 minutes

Required Document: JupiterMoons.ftm

Fathom Prerequisites: Students should be able to

- Open a document
- Create attributes defined by formula
- Create graphs of two attributes
- Create a slider

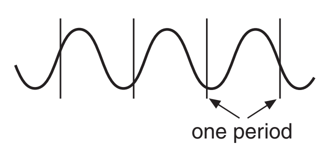
Fathom Skills: Students will learn how to

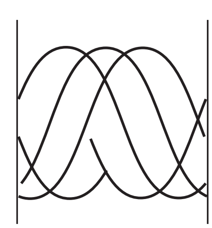
- Rescale a graph by using the pop-up menu to choose the graph type again
- Highlight points in graphs by selecting them in the case table
- Find the coordinates of a particular point in a graph
- Plot a function in a graph

Mathematics Prerequisites: Students should be able to write the equation for a translated and dilated periodic function.

Mathematics Skills: Students will learn how to transform data to improve a model; create a formula for the phase of a periodic function; write a periodic function to model data; and vary the period, phase, and amplitude of a periodic function to model data.

General Notes: This activity is about accuracy. How accurately can you determine how long it takes one of the moons of Jupiter to go around that planet? Students will propose one period and overlay the waves to see whether they line up. This requires a good understanding of the *phase* of a periodic function. Below left is a diagram of the path of a moon, with a proposed period—but the period is too short. If we overlay the periods, it looks like the diagram below right.





As students get the period closer and closer to being correct, the segments line up better and better. The nearer periods will line up first; the faraway ones need a very accurate period to line up, because even a tiny error is multiplied by the total number of periods. The concept of overlaying the cycles, once grasped, is powerful. And while the arithmetic is easy—and common in data processing—many advanced students have never come across a calculation like it.

Besides going over the questions on the student worksheet, ask students why the formula for *phase* works—and why it is needed. If your data cover a small range in *x* but their absolute values are large, then differences in slope (in a linear fit) create errors in the model that can overwhelm you, because tiny errors are magnified over that long distance from the *y*-axis. One solution is to translate the data back to the vertical axis.

INVESTIGATE

Q1 The points appear to be part of a trigonometric function. The period appears to be somewhere between 3 and 4 days. Estimates will be very rough at this stage.

Q2 The formula for *phase* should be

$$mjd - period \cdot floor\left(\frac{mjd}{period}\right)$$

Students may need help with this. If *mjd* were 163.5 and the period were 4.00, the *phase* would be 3.5, since 160.0 is exactly 40 cycles. Fathom uses *floor* for the next-lowest whole number, so *floor*(163.5 / 4.00) gives 40 for the number of whole cycles. Multiply the number of whole cycles by the period, then subtract from the current *mjd* to get the *phase.*

Q3 The behavior is very sporadic. It's hard to distinguish what is really happening. Students will learn how to modify *mjd* in step 6.

Q4 If students happen to adjust the slider to an integer multiple of the period, the points will also line up. Discuss this with students and make sure they look for the smallest value that lines up the points. It corresponds to January 1, 2000.

Q5 Students should get values within 0.3 day of those in Q6.

Q6 The estimates change slightly. *Io:* 1.771; *Europa:* 3.558; *Ganymede:* 7.169; *Callisto:* 16.839.

These estimates probably changed because there were more data to give the estimation more accuracy. Also, the new data points are far from the old points on the *x*-axis. This makes them more sensitive to small changes in the slider, improving the accuracy.

Q7 $\frac{\max(Europa) - \min(Europa)}{2}$

This function uses the maximum and minimum values of the data taken for *Europa* to get a value of 199.12. The data taken have gaps between values, so you can't be sure of the accuracy—the true min or max might lie in one of the gaps.

14. You may want to remind students about translation and dilation of a trigonometric function.

Q8 Sample answers:

$$Io = 126.73\sin\left(\frac{2\pi}{1.771}(phase - 0.55)\right)$$

$$Europa = -199.70\sin\left(\frac{2\pi}{3.558}(phase - 0.45)\right)$$

$$Ganymede = -310.5\sin\left(\frac{2\pi}{7.169}(phase - 0.24)\right)$$

$$Callisto = 548.7\sin\left(\frac{2\pi}{16.839}(phase + 2.17)\right)$$

Some students may have used cosine or used a different amplitude and displacement. This is a good opportunity for reviewing properties of periodic functions.

EXPLORE MORE

1. Students will need to rescale the graph to see the function clearly. The function should fit, because all that is changing is the unit of time.
2. Answers will vary. Check to make sure students calculated the date correctly.
3. Callisto has the longest orbital period, taking more than 16 days to get around Jupiter. Io has the shortest, just under 2 days.

POPULATION GROWTH (PAGE 63)

Activity Time: 50 minutes

Required Document: PopulationGrowth.ftm

Fathom Prerequisites: Students should be able to

- Open a document
- Create attributes defined by formula
- Create graphs of two attributes

Fathom Skills: Students will learn how to

- Create new cases
- Create and name sliders
- Plot a function
- Change slider bounds
- Change the scale of the graph

Mathematics Prerequisites: Students should be familiar with exponential and logistic functions. (This activity can also serve as an introduction to logistic functions.)

Mathematics Skills: Students will learn how to model population with an exponential function and modify the function to take limiting factors into account, creating a logistic model.

General Notes: In this activity, students take a simple model for population, and gradually modify it to reflect various types of changes in the population. They fit a function to the data, making sure that the function fits the data at several scales. This activity could also be appropriate for Algebra 2 students studying exponential models.

EXPERIMENT

Q1 *year* represents the number of years after you moved to Chelmsdale. *new* represents the number of new people that have come to live in Chelmsdale that year. *pop* represents the total number of people in Chelmsdale.

Q2 The population is 4882.81. The quick rise in population could be due to any number of occurrences. For example, a large corporation may have moved to town, bringing many jobs; families may be having more children; or there may have been recent housing development in town.

Q3 Exponential function. (Some students may mistake this for a power function.)

Q4 After 24 years, this population would be 423,516 according to our model. It does not seem reasonable for the town to be this large after barely one generation. This is the first hint of the limiting factors students will explore later in the activity.

INVESTIGATE

Q5 As you increase A, the function becomes steeper (the population increases more rapidly). When $A = 0$, the function is a horizontal line at $pop = 0$. When A is negative, the function is below the x-axis.

Q6 As you increase B when $B > 1$, the function becomes steeper (the population increases more rapidly). When $B = 1$ or $B = 0$, the function is flat. When $B < 0$, the function is undefined. When $0 < B < 1$, the function grows as *year* becomes more *negative*, indicating that for this model to represent growth, B must be greater than 1.

Q7 $A = 2000$, the initial population of Chelmsdale. $B = 1.25$, the rate of population growth. (Students should understand that B must be greater than 1 to have the population increase.)

Q8 You must change 0.25 to 0.27. This is because there must be more new people coming into the community than the original model expressed.

Q9 The new formula subtracts the deaths of the previous year and should read

$$\text{if } (caseindex = 1)\begin{cases} 2000 \\ prev(pop) + prev(new) - prev(deaths) \end{cases}$$

The exponential equation for *pop* still fits.

Q10 $pop = A(1 + newrate - deathrate)^{year}$

Q11 As the population grows, the rate of death also grows. (This question hints at the concept of derivative.) Students may notice that the graph points are “clumping” as the population gets close to its limit.

Q12 As time goes on, the population seems to be stabilizing at a specific number, which appears to be 13,500.

Q13 When *newrate* = 0.28, the population appears to approach 14,000. When *newrate* = 0.29, the population appears to approach 14,500. For each percentage point increase, the population limit appears to increase by 500 people.

EXPLORE MORE

Assuming students use 13,500 for c, the values are $a = 5.75$ and $b \approx 1.29$, giving

$$pop = \frac{13{,}500}{1 + 5.75(1.29)^{-year}}$$

This function should be a good model for their data.

EXTENSION

Have students pick two cities, research their growth, and model the data using Fathom.

RATES OF CHANGE (PAGE 67)

Activity Time: 30 minutes

Required Document: RatesOfChange.ftm

Fathom Prerequisites: Students should be able to

- Open a document
- Work with sliders and formulas
- Create graphs

Fathom Skills: Students will learn how to

- Enter formulas for attributes in case tables and graphs
- Manipulate sliders and graphs

Mathematics Prerequisites: Students should be familiar with slopes (rates of change) of functions and have been introduced to the concepts of derivative and instantaneous rate of change.

Mathematics Skills: Students will get a better understanding of the derivative by comparison of the average and instantaneous rates of change.

General Notes: Students will explore the difference between the average and instantaneous rate of change both numerically and graphically. The activity concentrates on power functions.

EXPERIMENT

Q1 a is the derivative. You could describe this kind of motion as a racecar, initially at rest, that picks up speed and accelerates faster as time passes.

Q2 v is an increasing linear function; a is a horizontal line at $a = 1$.

INVESTIGATE

Q3 As x increases, the difference between a and *Ave_Rate* gets larger. $x = 5$, Difference = 14; $x = 20$, Difference = 59; $x = 35$, Difference = 104.

Q4 Difference = 7.25. The difference will continue to decrease as n decreases.

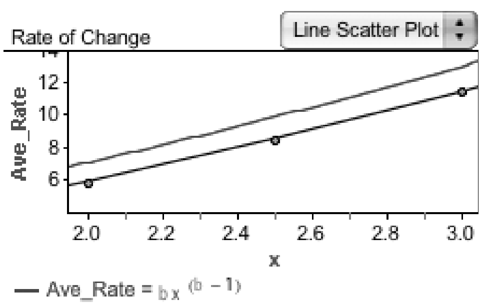
Q5 $x = 5$, $a = 75$

n	*Ave_Rate*	$a -$ *Ave_Rate*
1	61	14
0.5	67.75	7.25
0.2	72.04	2.96
0.01	73.51	1.49
0.05	74.25	0.75
0.01	74.85	0.15

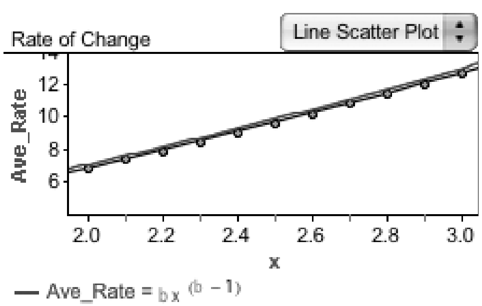
Q6 The instantaneous rate of change at $x = 5$ is the limit of the average rate of change at $x = 5$ as the interval approaches 0.

Q7 The *Ave_Rate* curve gets closer and closer to the actual derivative as n gets smaller.

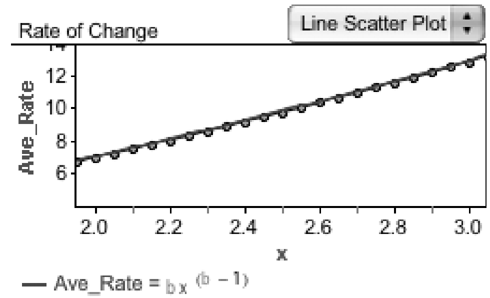
$n = 0.5$



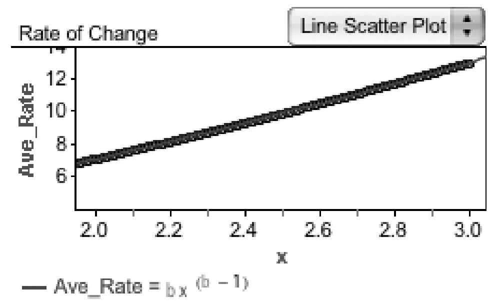
$n = 0.1$



$n = 0.05$



$n = 0.01$



Q8 $f(x) = v$; In questions Q3–Q6, $c = 5$. In Q7, c is any of the points on the *Ave_Rate* graph. The derivative is the limit of the average rate of change over the interval from c to x as x approaches c.