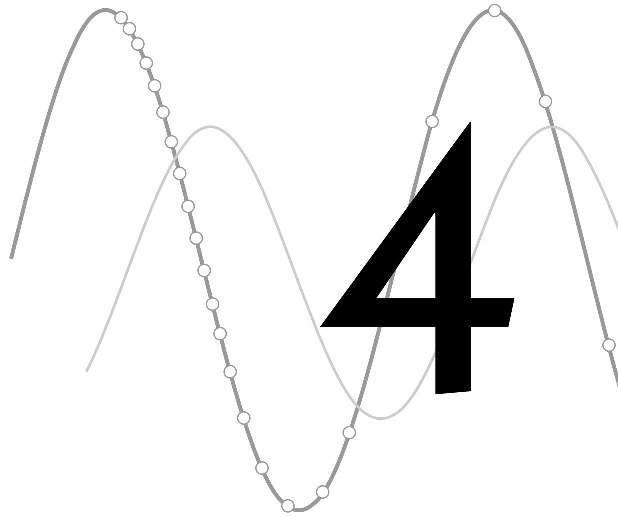
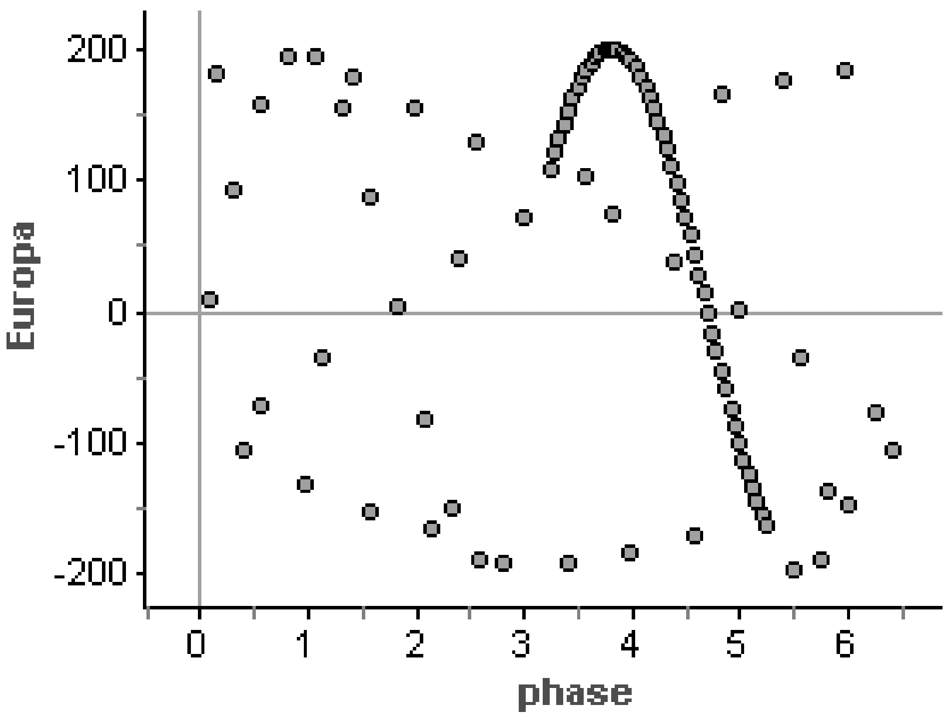


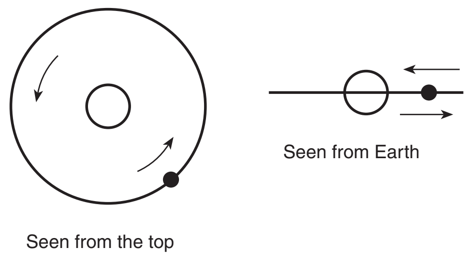
Precalculus and Calculus



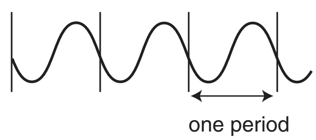
Jupiter's Moons

Astronomers have been able to model many parts of the universe to accurately predict where planets, stars, comets, and moons will be at any time. In this activity, you'll look at data about one of Jupiter's four largest moons and estimate how long it will take for the moon to complete one full revolution around the planet.

If you look at one of Jupiter's moons from night to night, it seems to be swinging back and forth. It's really going around the planet, of course, but you're seeing the moon's orbit edge-on.



If you record the distance of the moon from Jupiter—positive to the west, negative to the east—as a function of time, you get a wavy curve. You will find the period of this curve.

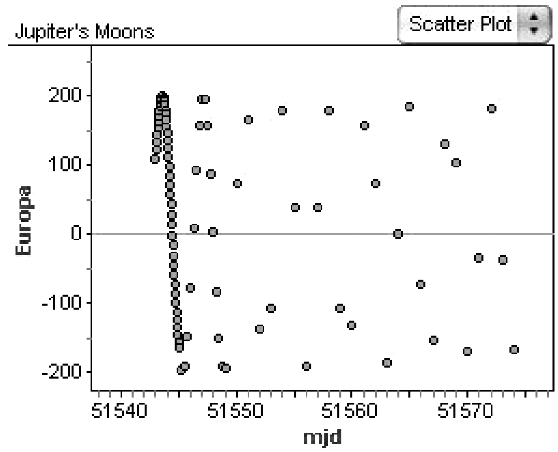


INVESTIGATE

1. Open the document **JupiterMoons.ftm.** You should see a case table with ten attributes.

The attribute *mjd* stands for Modified Julian Date in decimal days. (The Modified Julian Date measures days since November 17, 1858.) The attributes for the four moons, *Io*, *Europa, Ganymede,* and *Callisto,* give the distance of the moon from Jupiter as seen from Earth, measured in arc seconds $\left(1 \text{ arc second} = \frac{1}{3600} \text{ degree}\right)$. Positive distances are to the west; negative distances are to the east.

2. Choose one of the moons to use in this activity. You'll be able to experiment with the other moons later. Create a graph of the distance of the moon you chose as a function of *mjd*. Examine the table to understand why the initial points in the graph are tightly clustered together.



Q1 What specific kind of periodic function does this graph appear to be? Estimate the period of the graph.

Now you'll attempt to find a function to model this graph.

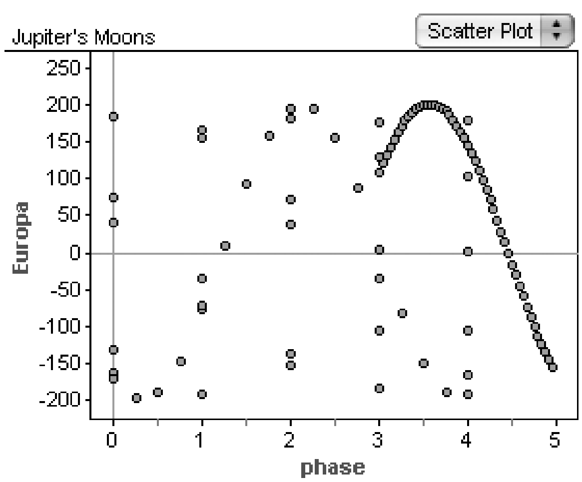
3. Create a new slider called *period*.

The *phase* of a periodic function is how far a function is into its current cycle.

4. In the case table, create a new attribute called *phase*. Enter a formula for *phase* that calculates the phase of the function using your slider for the period. (*Hint:* You want to find the time since the function has completed its last cycle. Use the *floor* function to help you find the number of whole cycles completed.)

Q2 What formula did you come up with?

5. Make a new graph that plots the distance of your moon as a function of *phase*.



Q3 Adjust the *period* slider. Describe what you see in the graph. What period seems to best fit the data for your moon? How did you choose that period?

The *mjd* values are so large that any changes in the estimate for the period lead to a large change in the calculation of the *phase*.

6. Add a new attribute, *newjd*. Edit the formula for this attribute to be *mjd* − 51544.

Q4 What date does the number 51544 correspond to in your model?

7. Edit the *phase* formula to use *newjd* instead of *mjd*.

8. Adjust the *period* slider until the points line up and show just a single cycle. (You may have to rescale the graph axes to see the complete cycle.) Try to be as accurate as possible by shortening the range of values for the slider. Do this by double-clicking on the slider axis and changing the lower and upper values. You will probably want to adjust the slider several times as you get more and more accurate.

Q5 What value of *period* to three decimal places makes the points in the graph line up? How accurate do you think your estimate is?

You now have an estimate for the period. But your data cover only a few cycles. Next you'll test your estimate by using data from several months later.

9. Add these data to your case table.

Date	Modified Julian Date	Io	Europa	Ganymede	Callisto
July 1	51726.0	27.28	−26.23	−239.64	−196.76
July 2	51727.0	14.45	−152.89	−83.11	−34.57

10. Select the new points by holding down the Shift key as you click on the case number for each new point in your case table. The points should be highlighted red in the second graph.

11. Now adjust the value of *period* to make the two highlighted points in the *phase* graph fit the curve.

Q6 Does your estimate for *period* change due to adding the new points? If so, what is your new estimate and why do you think it changed? Which estimate do you think is more accurate?

Now you'll create a function that models the data. You already know the period of the function (although the true period requires a bit of manipulation of the period you've already found). You'll need to come up with values for amplitude and phase displacement and decide whether to use the sine or cosine function.

12. Use your graphs to make an estimate for the amplitude of the function. Try using the *max* and *min* functions of Fathom to create a formula for the amplitude.

Q7 State the formula and the value you found. Explain why this is a good starting point for estimation, but probably isn't the actual amplitude of the function.

13. Before you can find the phase displacement, you need to decide which function to use. Although you can use either sine or cosine, one may be easier than the other. Once you have chosen which function to use, estimate the phase displacement by inspecting the graph. (*Hint:* When you click on a point on the graph, the coordinates of the point are shown in the bottom-left corner of the Fathom window. Think of points that may be helpful in estimating the displacement.)

14. You may need to adjust the amplitude and phase displacement, because they are only estimates. To help with adjustment, create two sliders—*amplitude* and *displacement.*

15. Now you have all the information for the equation. Click on the *phase* graph and then choose **Plot Function** from the **Graph** menu. Enter a function for the graph, using *amplitude*, *displacement,* and *period* as constants in your equation. Adjust the sliders to improve your model.

Q8 State the function that best matched your data. Could you write a different function that would model the data equally well? Explain.

EXPLORE MORE

1. Plot your function on the first graph you created, replacing *phase* with *mjd*. Does the graph fit the data? Explain.
2. Predict the position of the moon you chose at midnight on your birthday in 2006. Remember to convert to the New Modified Julian Date.
3. Compare your findings with those of other members of the class. Compare your graphs. Which planet has the longest orbital period? Can you determine which moon is closest to Jupiter? Which moon is farthest away?

Population Growth

Most populations are subject to a number of factors that affect the rate at which the population grows or decays. The rate at which a human population grows often depends on limiting factors, such as the availability of housing, food, and other resources. In the first three parts of this activity, you'll model a population assuming no limiting factors. In the last part, you'll introduce a limiting factor and study how this affects the population.

EXPERIMENT

Suppose you are part of the swiftly growing community of Chelmsdale. When you first moved to Chelmsdale, there were 2000 inhabitants, but in each of the next 4 years, Chelmsdale has grown at a rate of approximately 25%.

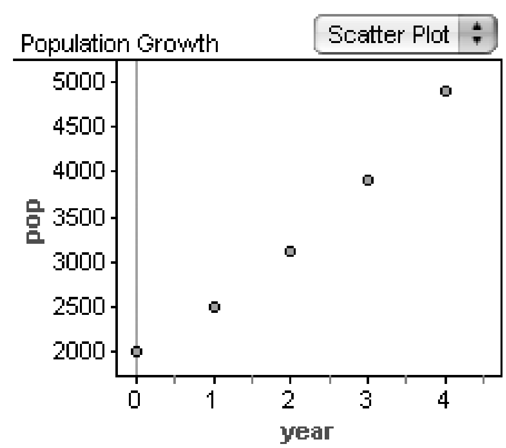
1. Open the document **PopulationGrowth.ftm.** You should see a case table with the attributes *year, new,* and *pop* and a formula for each attribute.
2. Select the case table. From the **Collection** menu, choose **New Cases.** Create five new cases.

Q1 What do the three attributes represent?

Q2 What is the population of Chelmsdale after 4 years? What could be some of the causes for the rise in population?

3. Graph *pop* as a function of *year.*

Q3 What kind of function have you graphed?



Now you will use your model to see how many people might live in Chelmsdale if the population continues to grow at this rate.

4. Select the table and add 20 new cases. The graph should rescale to show the new points. If necessary, resize the graph by dragging a corner.

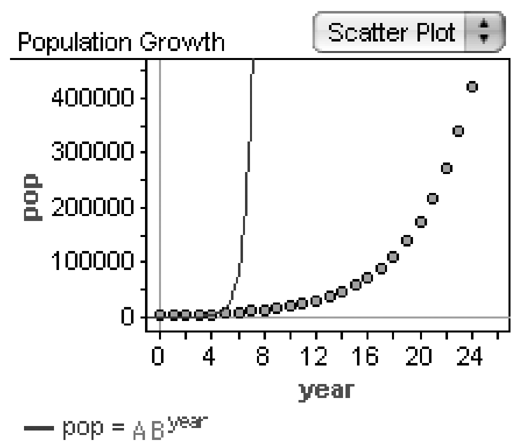
Q4 After 24 years, how many people live in Chelmsdale? Does it seem reasonable for the population to have grown this much in 24 years? Why or why not? (You'll explore this idea more later.)

INVESTIGATE

Now you'll find an exponential function that fits these data. You'll experiment using sliders.

5. Drag two sliders from the shelf into your document. Rename them *A* and *B* by clicking on their names and typing new ones.

6. Select the graph and choose **Plot Function** from the **Graph** menu. Enter the formula `A*B^year` in the formula editor. A curve should appear on the graph.



7. Adjust the *A* slider and watch how the curve changes. (You may want to change the bounds on the slider axis to bring larger numbers into view. Grab the right end of the axis and drag left.)

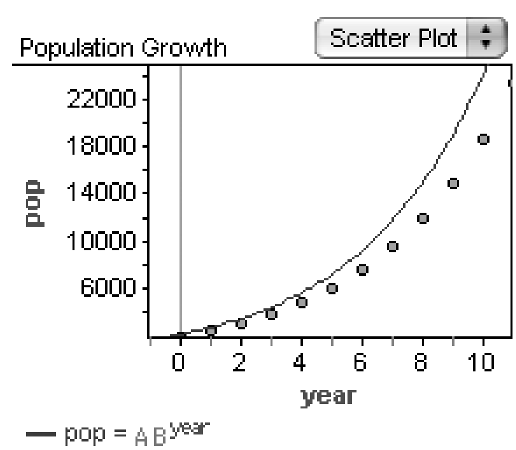
Q5 Describe the way the curve changes when you change the value of *A*.

8. Adjust the *B* slider. (The curve is very sensitive; you may want to restrict the values of *B* that you see. Double-click near the axis and enter new values for the *Lower* and *Upper* bounds.)

Q6 Describe the way the curve changes when you change the value of *B*.

The scale on the vertical axis is very large, and it's hard to see how well the function is fitting the smaller values.

9. Adjust the scale of the graph so that you can focus on the first 10 years. Double-click the graph to show its inspector. Change *xUpper* to 11. Pick values for *yLower* and *yUpper* that will show only the first 10 years in the graph.



10. Move both sliders to fit the function to the data. (*Hint:* The values of *A* and *B* have something to do with the original data you were given.) After you have matched the function to the data for the first 10 years, select the graph and choose **Rescale Graph Axes** from the **Graph** menu. If your function doesn't fit the later points, adjust the sliders until it does.

Q7 What values of *A* and *B* fit the data best? How did you come to this conclusion? Explain as clearly as you can what *A* and *B* represent.

You may have noticed a flaw that makes this simulation unrealistic—for many of the years, there are non-integer values of people. Next you will update the model to use only integer values of people.

11. Change the original formula for *new* so that there are only integer values of new people coming into the community. (Try using the *floor* function in the formula editor.)

12. The original model also neglected to include the death rate. In the case table, create a new attribute called *deaths*. Assume that the death rate is constant at 2% per year. Enter the formula for *deaths*. Don't forget to use integer values.

Q8 You know the death rate is constant. But the percentage increase in the actual population of Chelmsdale is known to be 25% each year. How must you change the formula for *new* to take into account the addition of *deaths* to your model? Explain.

Q9 Now change the formula for *pop* to take into account these changes. What is the new formula? Does it still agree with your exponential equation for *pop*?

Q10 See if you can express *B* in terms of the rate of new inhabitants and the death rate. Create two new sliders, *newrate* and *deathrate*, to take the place of *B*. Change your exponential equation to reflect these changes. What is your new equation?

13. Update the formulas for *pop*, *new*, and *deaths* in the case table by inserting the three sliders in place of the constants. Experiment with different values for the rate of new inhabitants, death rate, and initial population by moving your sliders.

Real populations do not grow without limit. There are a number of factors that can limit the growth of a population. For example, a community has only a certain amount of land to build new homes. Also, water, food, and energy resources are limited, thereby limiting the number of people that can come in to a community. This phenomenon is often called the *crowding effect*.

To simulate the crowding effect, you'll increase the death rate as the population of the community becomes too large. There are several ways to do this. One of the most interesting ways is to change the formula so that the death rate gradually increases with the population. That's what you'll do.

14. Edit your formula for *deaths* by multiplying *deathrate* by $\frac{pop}{1000}$. That way there will be fewer deaths when $pop < 1000$ and more deaths when $pop > 1000$. Set the *A* slider to 2000 and set the *newrate* and *deathrate* sliders to their original values.

Q11 Graph *deaths* as a function of *pop*. Describe what happens.

Q12 Rescale your graph of *pop* as a function of *year* by choosing **Rescale Graph Axes** from the **Graph** menu. You should have just created a *logistic function.* Describe the graph. Add 50 new cases to your case table and scroll down. What number does *pop* appear to be approaching?

Q13 Increase *newrate* to 0.28. You can type the value directly into the slider by clicking its current value. What number does *pop* approach now? What about when *newrate* is 0.29? Do you see a pattern?

EXPLORE MORE

The logistic function has the form $y = \frac{c}{1 + a(b)^{-x}}$,where *a, b,* and *c* are constants. The function has two horizontal asymptotes at $y = 0$ and $y = c$. Create a logistic function that models the population growth including the limiting factor. The population limit for your graph can be modeled with a horizontal asymptote. Use this limit as your value for *c.* Choose two points from the case table to solve for the constants *a* and *b.* Does it matter which two points you choose? Solve for *a* and *b* and write the resulting equation. Graph the equation of the logistic function on your graph of *pop* as a function of *year.* Is the function a good model for the population? Explain.

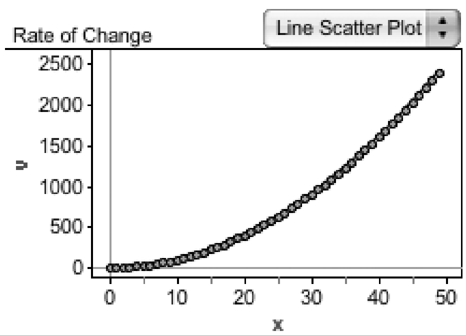
Rates of Change

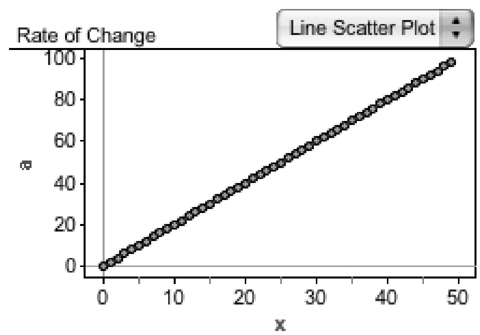
You may have learned that the connection between position (or distance), velocity, and acceleration has to do with derivatives. In this activity, you'll develop an initial understanding of the mathematical relationship between these concepts. You'll explore the difference between the average rate of change and instantaneous rate of change of a function.

EXPERIMENT

1. Open the document **RatesOfChange.ftm.** You should see an empty case table with attributes *x*, *v*, and *a*, two empty line scatter plots, and a slider, *b*. Attribute *x* stands for time, *v* stands for velocity, and *a* stands for acceleration.

2. Select the case table. Choose **New Cases** from the **Collection** menu, enter `50`, and click **OK.** You should see points on each of the graphs that look similar to the ones shown here.





Q1 Describe attribute *a* in terms of the derivative of *v*. (*Hint:* The derivative of a power function decreases the exponent by one.) What motion in the real world might have this velocity and acceleration?

3. The slider *b* represents the exponent of the power function $v = x^b$. Move the slider back and forth and observe the change in the graph of *v*.

Q2 Set *b* to 1. (You can do this directly by clicking on the value of *b* and typing in the new value.) Rescale each graph by selecting the graph and choosing **Rescale Graph Axes** from the **Graph** menu. (You can also do this by choosing **Line Scatter Plot** again from the pop-up menu in the graph.) What kinds of functions are shown on each of the graphs now? Be as specific as you can.

The *average* rate of change of a function calculates the slope between two points of a function. Next you will calculate the average rate of change of *v* and explore how it relates to the derivative.

4. Set b to 3.00 and rescale the graphs. Create a new attribute by clicking on <*new*> in the case table and entering `Ave_Rate`.
5. Select *Ave_Rate* and choose **Edit Formula** from the **Edit** menu. Enter the formula

```
(v-prev(v,""))/(x-prev(x,""))
```

and click **OK.** ("") means that if there is no previous value of v and x, no value of *Ave_Rate* will be calculated.

INVESTIGATE

In question Q1, you should have answered that a is the derivative of v. Another way to say derivative is *instantaneous* rate of change. So, you know the instantaneous rate of change of v is a. We want to compare *Ave_Rate* to a.

Q3 Compare the values for a and *Ave_Rate*. Do you see a trend in the difference between a and *Ave_Rate* as x gets larger? Calculate the difference when $x = 5$, $x = 20$, and $x = 35$.

Now, let's see what happens when we decrease the distance between x-values. We'll do this using a slider that directly affects the increment of the x attribute.

6. Drag a new slider from the object shelf and drop it into the document. Name the slider n. (Click the slider name and type in the new name.)
7. Next, insert the value of the slider into the formula for x. Select the x attribute in the case table and choose **Edit Formula** from the **Edit** menu. Change the formula to `n(caseindex-1)`.
8. Finally, you want x-values to be a fraction of what they were, so you need to limit the values of the slider to account for this. Double-click the slider axis. Set the lower value to 0 and the upper value to 1, then close the inspector.
9. Check to make sure you have done this correctly by setting $n = 1$. The values for x should be the same as they were before you changed the function. If they aren't, go back and redo steps 6–8.

Q4 Set $n = 0.5$. Find the difference between a and *Ave_Rate* at $x = 5$. Is the difference less than when $n = 5$? Describe what you think will happen as you continue to decrease the value of n.

Q5 Repeat Q4 for $n = 0.2, 0.1, 0.05, 0.01$, and 0.001. $\left(\text{You will have to add more cases each time. To find the number of cases you need, calculate } \frac{5}{n}.\right)$ Record the values you get for *Ave_Rate* and $a - $ *Ave_Rate* in a table. What conclusions do you draw?

Q6 Based on what you've just discovered, define the instantaneous rate of change at $x = 5$ in terms of the average rate of change. (*Hint:* Use the limit of the average rate.)

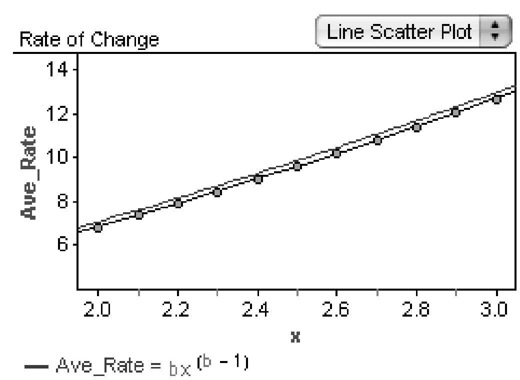
You've compared average and instantaneous rate of change numerically. Next, you'll approach the same problem graphically. You'll observe graphs of the derivative and the average rate of change as the increment of x decreases.

10. For a change of pace, set $b = 2.5$. Set $n = 1$. If you have more than 300 cases in your case table, delete the extra cases. To delete cases, select them and choose **Delete Cases** from the **Edit** menu.

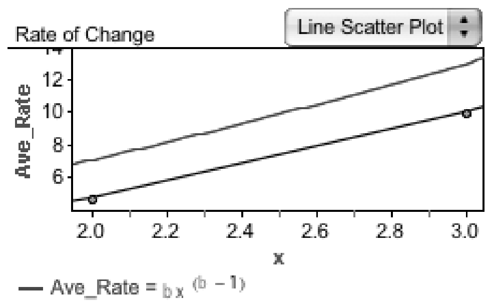
11. Drag a new graph into the document. Drag attribute x from the case table and drop it below the horizontal axis of your new graph. Drag *Ave_Rate* to the vertical axis. A scatter plot should appear. Choose **Line Scatter Plot** from the pop-up menu on the graph.

12. You want to compare this to the derivative (or instantaneous rate of change), so let's put a graph of the derivative on the same graph. Select the a attribute in the case table and choose **Copy Formula** from the **Edit** menu. This is the formula for the derivative.

13. Now select the graph you just created and choose **Plot Function** from the **Graph** menu. Paste the formula here and click **OK.** The two curves should look almost identical.



14. Let's zoom in on a small portion of the graph, to make it easier to see what happens as the increment between x-values gets smaller. To zoom in on a smaller portion of the graph, double-click the graph to show the graph inspector. Change the bounds to $1.95 \leq x \leq 3.05$ and $4 \leq y \leq 15$. Also, change *xAutoRescale* and *yAutoRescale* from true to false. (This keeps the scale of the axes the same as you change the data.) The graph should look similar to the one shown here.



Q7 Experiment with different values of n. Try $n = 0.5, 0.1, 0.05$, and 0.01. (As n decreases, you may want to make the axes bounds smaller to see just how close they get.) Describe what happens on the graph as n gets smaller.

Q8 The formal definition of derivative of a function, $f(x)$, at point $x = c$ is the instantaneous rate of change of $f(x)$ with respect to x at $x = c$. Describe $f(x)$ and c in the context of this activity. Can you come up with a numerical definition of derivative?

EXPLORE MORE

Try repeating the activity with different values for b.