that the number of differences necessary to get constant values is always the same as the degree.

Now ask, "Assume you are given any sequence of numbers. How could you use what you've learned to determine whether the sequence is modeled by a polynomial function? How would you determine the degree of the function?"

THE CIRCUMFERENCE FUNCTION (PAGE 25)

Activity Time: 30–45 minutes

Materials:

- Many circular objects of different sizes
- Centimeter rulers or measuring tapes
- String

Fathom Prerequisites: none

Fathom Skills: Students will learn how to

- Create a collection by entering data into a case table
- Create attributes with formulas
- Create graphs (scatter plots)
- Graph lines of fit (movable lines)
- Use summary tables to calculate measures of center (mean)

Mathematics Prerequisites: Students should be able to make a scatter plot of points; find the slope of a line; and write the equation of a line in slope-intercept form.

Mathematics Skills: Students will learn how to approximate the value of π by analyzing the relationship between diameter and circumference; think about π as a functional relationship between diameter and circumference and as a ratio of circumference to diameter; find a direct-variation linear equation to model data that are roughly linear; and interpret the slope of a direct-variation linear equation as a constant ratio between the dependent and independent variables.

General Notes: Many geometry teachers already use an activity in which students measure the circumferences and diameters of circular objects and "discover" that the ratio is π. This is the same activity enhanced by an analysis in Fathom. By using Fathom to look at the data with tables, graphs, proportions, and summary statistics, students can get multiple representations of the same concept and gain a deeper understanding of π.

The activity opens by modeling the scientific method—challenging students to make a conjecture before they collect data. Because π is frequently taught in mathematics long before high school, many students will already know that the relationship is $C = \pi d$. Have those students answer questions Q1 and Q2 based on the relationship that they know to be true, but also ask them to conjecture how their experimental measurements might compare to the theoretical formula.

Students then collect data by physically measuring circular objects. You may want to plan ahead and ask every student to bring in two or three circular objects that the entire class can work with. If students don't bring in objects and you don't have access to enough circular objects, use geometry software or a compass to make and cut out a variety of circles. While students are making measurements, make sure that they are using measuring tapes or string to directly measure the circumferences. If they measure the diameter and then calculate the circumference, they will see a proportional relationship, but they will have missed the point! To save class time, you might want to have students collect measurements at home, rather than actually bringing objects to class.

The analysis of the data begins with a scatter plot and a line of fit. You could jump directly to calculating the ratio of circumference to diameter and finding a measure of center. However, jumping immediately to ratios makes the assumption that the relationship is a direct variation, and it doesn't tell you whether the same ratio applies to circles of all sizes. Furthermore, a ratio of two variables is, by its nature, merely a summary. A scatter plot helps you see not only the relationship between the variables, but also whether the relationship continues. And because the scatter plot shows a separate point for each case, it shows all of the information and allows you to make predictions by interpolating and extrapolating. Using a scatter plot is a particularly important first step if you do Explore More 2, in which calculating a ratio alone obscures the importance of the intercept.

While students are fitting lines to the data, you may want to emphasize the functional relationship between *Diameter* and *Circumference.* Review that a function is a relationship in which each input value results in one output value. The students' scatter plots of *Circumference* versus *Diameter* probably pass the vertical line test, and the lines of fit definitely pass the vertical line test. You could also point out that the relationship is actually one-to-one: each input value results in one and only one output value. That is, *Circumference* is a function of *Diameter,* and *Diameter* is a function of *Circumference.* You could illustrate this relationship by swapping the axes in the scatter plot and showing that the resulting graph still passes the vertical line test.

After students have fit direct-variation lines to their scatter plots and seen that the relationship between *Circumference* and *Diameter* is represented by the slope, they go on to the "traditional" method of calculating the ratio of *Circumference* to *Diameter.* The activity uses a summary table to calculate the mean. If your students have enough cases, you might also have them create one-variable graphs (dot plot, histogram, or box plot) to see how the "average" is graphically represented by clusters or mounding. Students could also add other measures of center, such as the median, to the summary table and discuss pros and cons of each summary.

For the sake of geometry students who have never used Fathom before, the worksheet provides ample Fathom instructions. If you are using this activity with novice students, please allow sufficient time for students to learn both the software and the mathematics. On the other hand, if your students are very familiar with Fathom, you can expect the activity to move at a fairly rapid pace. You might even prefer to set up the scenario but omit the worksheet, letting students make their own conjectures, design their own data-collection methods, and do their own Fathom analysis.

MAKE A CONJECTURE

Q1 Students who are familiar with the relationship might say, "The circumference is about 3.14 times the diameter," or state the formula $C = \pi d$.

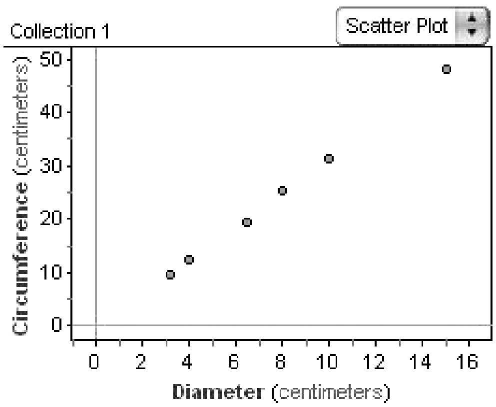
Q2 Although many students may be familiar with the formula $C = \pi d$, they may never have thought of it as a linear equation with slope π and intercept 0.

EXPERIMENT

Q3 Students will probably use mental math to divide each circumference by the corresponding diameter. They might estimate each ratio to be about 3, which probably supports the conjecture in Q1.

INVESTIGATE

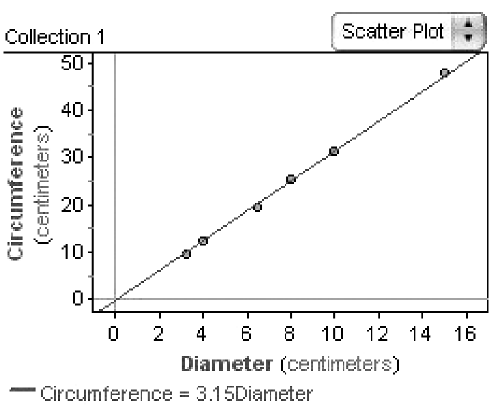
Q4 The points should be roughly linear with intercept 0. The slope should be about 3. Answers about whether or not the graph supports a student's conjecture in Q2 will vary.



Q5 The pattern is linear, so a line would fit the points.

Q6 Yes, the line should go through (0, 0), because a circle with no diameter would have no circumference.

Q7 The linear equation should be approximately *Circumference* = 3.14*Diameter*. The number (coefficient) multiplied by *Diameter* represents the relationship between the diameter and the circumference. This number is represented by the physical slope of the line.



Q8 By definition, a measurement is an approximation, not an exact value. So, students' measurements are likely to vary a little from the actual relationship. Because the points are so close to being linear, students will likely guess that the relationship is truly linear.

Q9 The ratios should all be around 3.14. Compared to the rest of the activity, the ratios should be close to the mental math calculations in Q3 and the slope of the line in Q7. *Note:* You may want to have a class discussion about why *CircDiam* has no units.

	Diameter	Circumference	CircDiam
units	centimeters	centimeters	
1	4.0 cm	12.4 cm	3.1
2	8.0 cm	25.2 cm	3.15
3	6.5 cm	19.5 cm	3
4	10.0 cm	31.4 cm	3.14
5	3.2 cm	9.6 cm	3
6	15.0 cm	48.0 cm	3.2

Q10 The mean should be around 3.14.

CircDiam	3.0983333

S1 = mean ( )

Q11 The circumference is about 3.14 times the diameter, or $C = \pi d$. This may or may not be what students conjectured in Q1. Students should make some observation about the variability between the experimental measurements and the theoretical formula.

Q12 Although many students may know the formula $C = \pi d$, they may never have thought of it as a linear equation. So, seeing the points form a line with slope π may surprise some students.

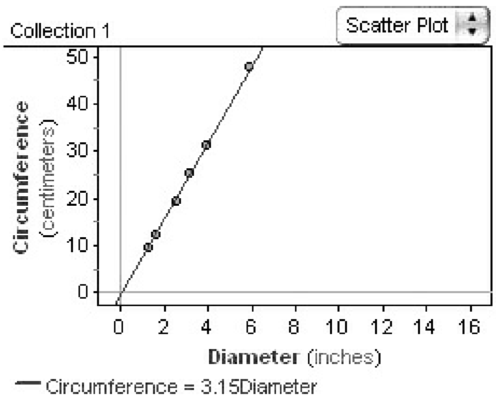
EXPLORE MORE

1. Measuring diameter and circumference in inches makes no difference—the ratio will still be π. Measuring in different units technically introduces the ratio $\frac{2.54 \text{ cm}}{1 \text{ in.}}$. That is,

$$\textit{Circumference} \text{ (cm)} = 2.54\pi \cdot \textit{Diameter} \text{ (in.)}$$

$$\textit{Circumference} \text{ (in.)} = \frac{1}{2.54}\pi \cdot \textit{Diameter} \text{ (cm)}$$

However, because Fathom has built-in unit conversion, the equation for the line of fit will appear not to change, although the graphical slope of the line does. The following graph shows that the physical slope of the points increases to about 8 when *Diameter* is measured in inches. But, because Fathom does unit conversions, the values of the attribute *CircDiam* are still about 3.14 and the equation of the line of fit still shows a coefficient of 3.14.



Collection 1

	Diameter	Circumference	CircDiam
units	inches	centimeters	
1	1.6 in	12.4 cm	3.1
2	3.1 in	25.2 cm	3.15
3	2.6 in	19.5 cm	3
4	3.9 in	31.4 cm	3.14
5	1.3 in	9.6 cm	3
6	5.9 in	48.0 cm	3.2

Some students might be confused why the graph shows a slope of about 8, yet the equation shows a slope of about 3.14. It's because the graph is dependent on the units used on the axes, while the equation shows a relationship between two distances that should have equal units. In other words, the equation shows that the relationship between circumference and diameter is always π, regardless of what units you use to measure.

2. Because students waste some amount of string while tying the knot, the relationship will not be the direct variation $Length = \pi \cdot Diameter$. If each knot requires 2 cm of string, then the relationship will be $Length - 2 = \pi \cdot Diameter$, or $Length = \pi \cdot Diameter + 2$. So, when students make a scatter plot and fit a line, the line will still have slope π, but will now have an intercept that represents the amount of string wasted by the knot. Because the relationship is not a direct variation, calculating the ratio *CircDiam* is meaningless in this scenario.

AREA AND PERIMETER (PAGE 29)

Activity Time: 40–55 minutes

Fathom Prerequisites: Students should be able to

- Create a collection by entering data into a case table
- Define attributes with formulas
- Make two-variable graphs (scatter plots)

Fathom Skills: Students will learn how to

- Add cases to a case table without adding data
- Use random numbers to simulate data
- Rerandomize data
- Make one-variable graphs (dot plots, histograms)
- Plot functions
- Create a slider and use its value as a parameter

Mathematics Prerequisites: Students should be able to calculate the area and perimeter of a rectangle; make a scatter plot of points; and graph linear and nonlinear functions.

Mathematics Skills: Students will learn how to simulate an experiment using technology; describe the distribution of one-variable data; fit curves (specifically, a parabola) to data; use algebra to mathematically explain a geometric situation; and see boundaries (limits) from a graph.

General Notes: In this activity, students simulate random rectangles and look for relationships between length, width, area, and perimeter. The activity takes students significantly deeper than the customary formulas $A = lw$ and $P = 2(l + w)$. It exposes students to statistics, algebraic modeling, and limits, and shows how these diverse areas of mathematics can be used to support each other. The activity also touches on the concept of optimization—How do you maximize area without maximizing perimeter?—which has many real-world applications, such as packaging design and manufacturing. That is, it illuminates the geometric concept that a square maximizes area for any perimeter.

As presented on the student worksheet, students use random numbers to simulate creating random rectangles. Before jumping into the Fathom activity, you may prefer to have students draw rectangles by hand, cut them out, and measure the lengths and widths. A quick way to do this is to give each student a piece of 8.5-by-11-inch paper and ask them to make three straight cuts to form three random rectangles. If you make a lot of rectangles as a class (say, 100 or more), you could enter your real data into Fathom and analyze the data in addition to or as a replacement for the computer simulation.

When students make the scatter plot of *Area* versus *Perimeter* in step 9, make sure that they understand what the position of each point represents in terms of a physical rectangle. Ask questions such as “What do rectangles in the lower-left look like?” (small in both dimensions) or “What do rectangles in the lower-right look like?” (long and skinny). Although the student worksheet eventually asks, “What type of rectangles are [along the curved boundary]?,” it is worth asking this at step 9 as well, to make sure that students see them as squares.

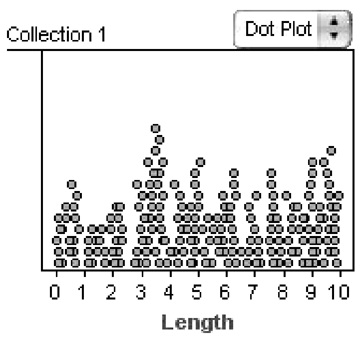
A challenging discussion question is “Why do you suppose so many of the points are close to the curved boundary?” Large-area rectangles must be close to being squares; otherwise, they’d have a dimension larger than 10. For smaller-area rectangles, you can have a surprisingly large length-to-width ratio before the rectangle moves a long way from the border. Statistically speaking, you should get more squarish rectangles because the distribution of the quotient of two uniform random variables, larger to smaller, is greatly skewed toward 1. In this case, that means the majority of values for *Length* $\div$ *Width* will be close to 1; so, the majority of rectangles will have about the same length as width, or be close to a square.

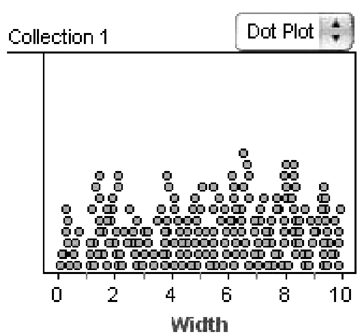
Although the context of this activity is geometric, it is also an excellent activity for algebra, probability, and statistics. Algebra and Precalculus students can practice creating scatter plots, fitting linear and quadratic functions, identifying limits, and algebraically proving why a particular function fits a set of data. Statistics students can practice using random numbers, performing simulations, comparing experimental versus theoretical probability, generating frequency distributions, and describing (and justifying) the shape and spread of distributions.

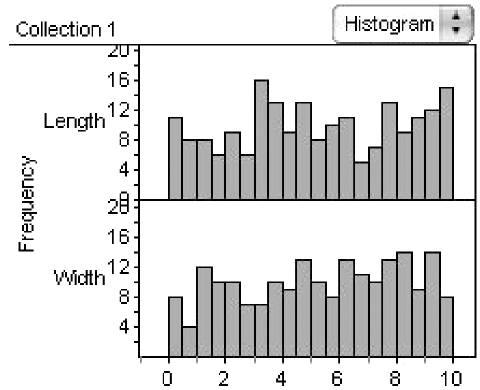
EXPERIMENT

Q1 The formula *random*(10) creates random numbers between 0 and 10; *random*(100) would create random numbers between 0 and 100; *random*( ) would create random numbers between 0 and 1. In general, the function *random* creates a random number between 0 and 1, which is then multiplied by the number in parentheses.

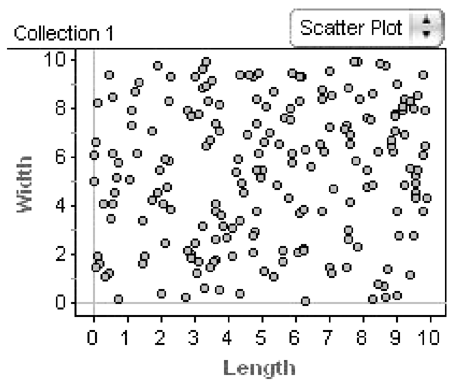
Q2 Technically these distributions are called *rectangular*. Because each random number between 0 and 10 has the same probability, stacks or bins of about the same height are created. (*Note:* Rather than making two separate graphs, students familiar with Fathom's ability to put multiple attributes on one axis might combine the dot plots or histograms, as shown in the third graph.)







Q3 Students should somehow mention that the scatter plot is a random arrangement of points between (0, 0) and (10, 10).



Q4 Students should use the formulas from the introduction to the worksheet: *Area* = *Length* • *Width* and *Perimeter* = 2(*Length* + *Width*). Students should verify the formulas by showing hand calculations. For the first case in the table below, 9.42286 • 5.50248 = 51.8491 and 2(9.42286 + 5.50248) = 29.8507.

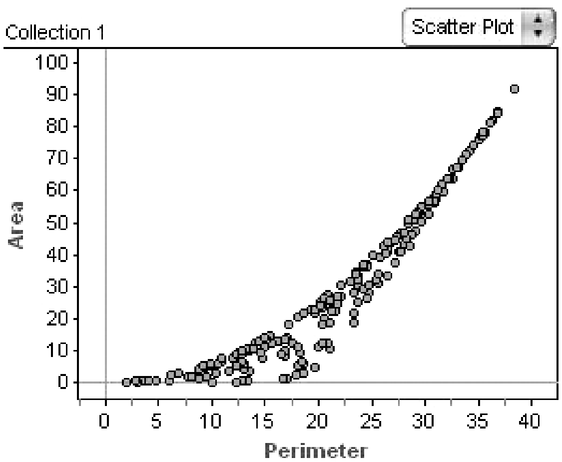
	Length	Width	Area	Perimeter
1	9.42286	5.50248	51.8491	29.8507

MAKE A CONJECTURE

Q5 Answers will vary.

INVESTIGATE

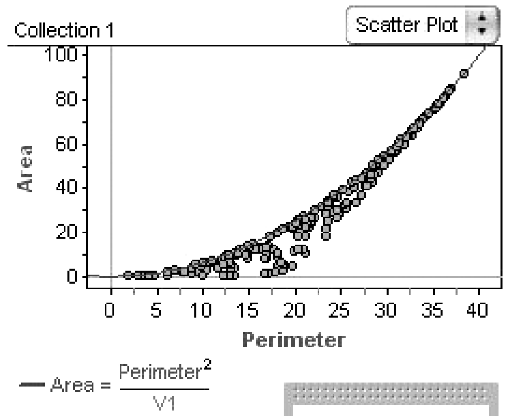
Q6 The actual scatter plot has a curved upper boundary and a linear lower boundary, as shown on the next page. This probably is not what students expected, especially after having seen *Width* versus *Length,* which had no discernible shape. Students will probably be surprised to see that the points form such clear boundaries. Even seasoned geometry students who know that squares maximize area and who might expect the parabolic curve may be surprised by the linear lower boundary.

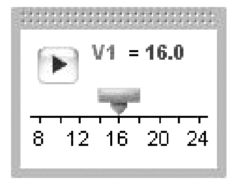


Q7 There are several ways to explain why there are no points in the upper-left region. An intuitive approach is to say that such a rectangle would have a big area and a small perimeter, which is impossible. For a given perimeter there's only so much area you can enclose (and some students may already know that a square maximizes area). For example, with a perimeter of 20 units, the most area you can enclose is a 5-by-5 square, or 25 square units. Therefore, the region above that limit is empty. Or, for a given area there's a minimum perimeter that encloses it. For example, with an area of 25 square units, 20 units is the smallest perimeter possible.

Q8 If students try a number between 0 and 1, the curve will get steeper, so they need to try larger numbers. If students try a number between 1 and 15, the curve will get shallower, but they'll still need to try a larger number. A few students might try 16, which will fit perfectly. If students try numbers greater than 16, they'll overshoot the boundary and will need to try smaller numbers. Students probably won't think to try negative numbers, but if they do, the graph will be reflected across the horizontal axis.

Q9 $V1 = 16$ makes a perfect fit.





Q10 The rectangles near the boundary have (approximately) the same length and width, which means they are squares or almost squares. Squares form the upper boundary because squares maximize area for any given perimeter.

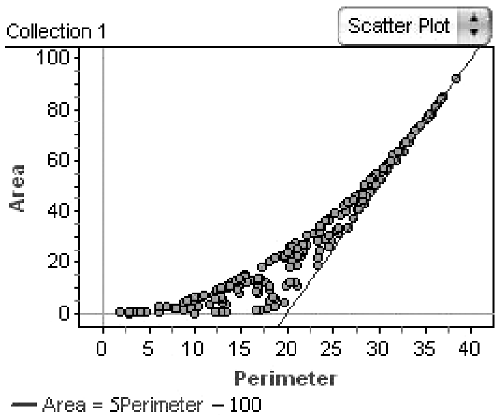
Note: You'll probably want to discuss how Q10 is directly related to Q7.

Q11 The area of a square is $Area = Length^2$. The perimeter of a square is $Perimeter = 4 \cdot Length$, or $Length = \frac{Perimeter}{4}$. Substituting for *Length* gives $Area = \left(\frac{Perimeter}{4}\right)^2$, or $Area = \frac{Perimeter^2}{16}$.

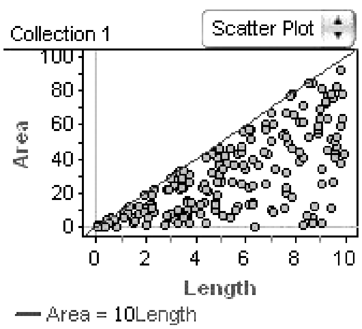
EXPLORE MORE

1. The empty region below the points exists because the maximum values of *Length* and *Width* limit the sizes of the rectangles. For example, if *Perimeter* were 30 units, then *Area* could not be 0 square units because then either *Length* or *Width* would have to be 15 units, yet the maximum for either is 10 units. This means there is a minimum area for any given perimeter. To minimize *Area* for a given *Perimeter,* two of the sides (say, *Length*) have to be as long as possible, 10 units each and 20 units combined, and the other two sides (say, *Width*) have to be as short as possible, or $Width = \frac{Perimeter - 20}{2}$. Then the

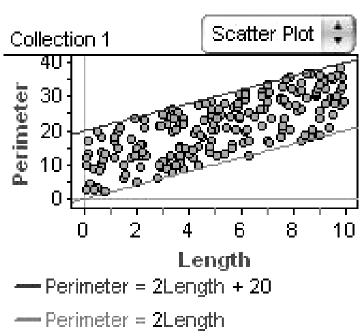
minimum area is *Area* = *Length* • *Width* = 10 • $\frac{Perimeter - 20}{2}$ = 5 • *Perimeter* − 100. Students could also find this linear equation by using a movable line:



2. *Area* versus *Length* has one linear upper boundary: *Area* = 10 • *Length*. This maximum occurs when *Width* is as large as possible, or 10 units.



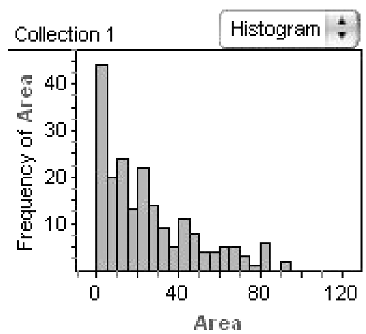
Perimeter versus *Length* has a linear upper boundary and a linear lower boundary: *Perimeter* = 2 • *Length* + 20 and *Perimeter* = 2 • *Length*. The maximum again occurs when *Width* is as large as possible, or 10 units. The minimum occurs when *Width* is as small as possible and can still make a rectangle, or very close to 0 units.



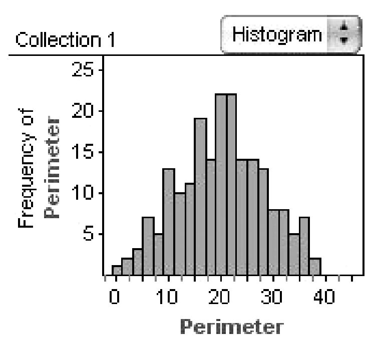
EXTENSIONS

1. Have students look at histograms of *Area* and *Perimeter,* describe the shape of these graphs, and explain why these shapes make sense. This is a particularly good extension for Statistics students.

   The graph of *Area* is skewed right, which means that smaller areas are more likely than larger areas. This makes sense because area is a product, and to get a large product you need two large factors; but because the length and width are randomly generated, it is less likely to get two large numbers paired together. As a simple example, imagine that length and width could each be 1 or 10. Then 1 • 1 = 1 (small), 1 • 10 = 10 (small), 10 • 1 = 10 (small), and 10 • 10 = 100 (large).



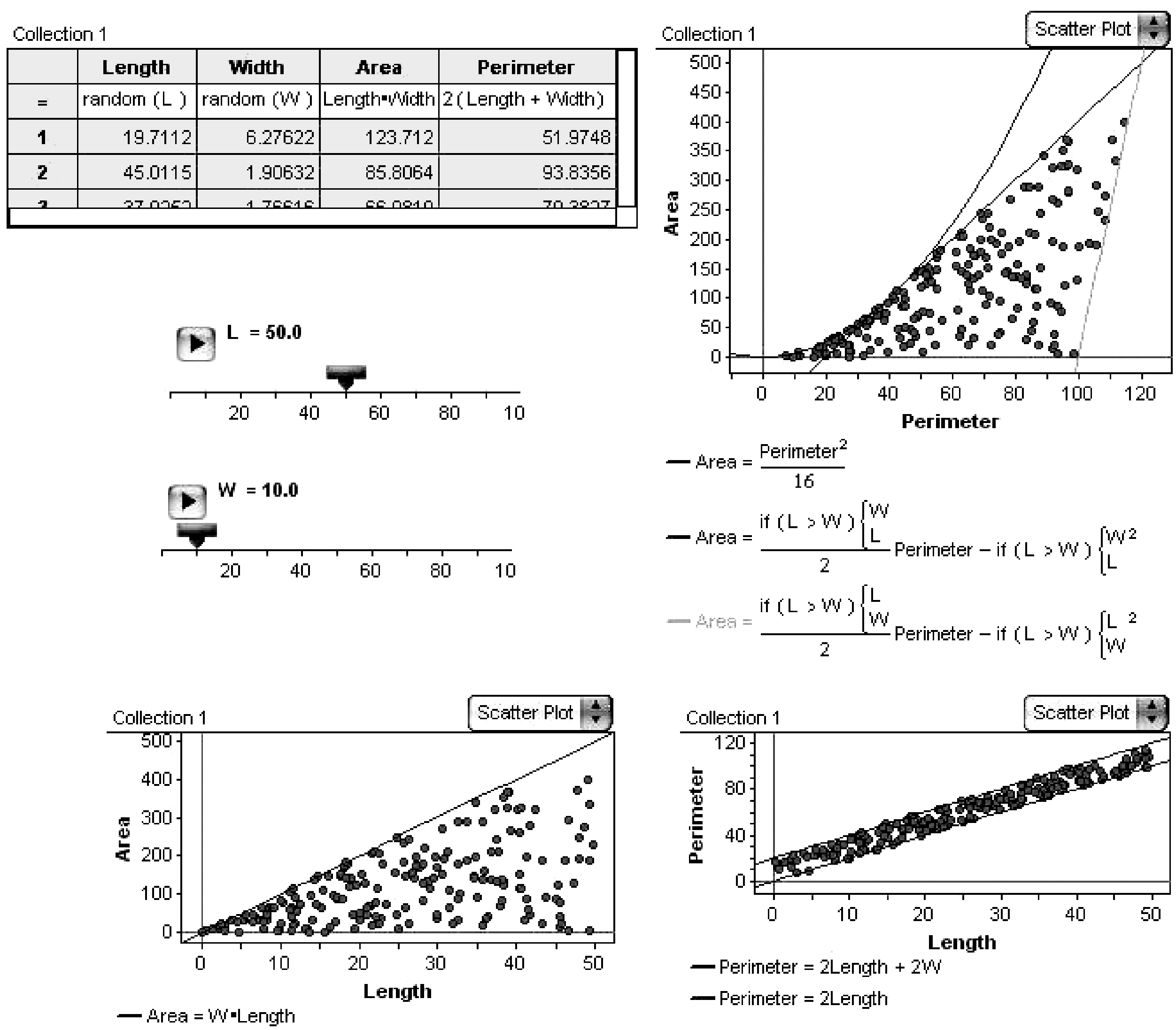
   The graph of *Perimeter* is triangular, which means that perimeters in the middle (between 15 and 25 units) are more likely than small or large perimeters. This makes sense because perimeter is a sum, and to get a small or large sum you need two small or large addends. Imagine the same simple example: 2(1 + 1) = 4 (small), 2(1 + 10) = 22 (medium), 2(10 + 1) = 22 (medium), and 2(10 + 10) = 40 (large).



After students have created histograms of *Area* and *Perimeter,* have them select any one bin in either histogram. Have them observe where the points are in the scatter plots of *Length* versus *Width* and *Area* versus *Perimeter.* (If they did Explore More 2, also look at the location of points in *Area* versus *Length* and *Perimeter* versus *Length.*) Ask students to describe the patterns that they see as they gradually select consecutive bins in the histogram. Challenge them to explain why these patterns exist.

2. Have students explore what happens when they change the formulas for *Length* and *Width* to a different random number, say, r*andom*(5) or r*andom*(100), or what happens when they are different formulas, say, *Length* = *random*(10) and *Width* = *random*(5). Ask students to first predict how the scatter plot of *Area* versus *Perimeter* and its boundaries would change, then actually try it in Fathom. Challenge students to generalize the results to describe what happens for *Length* = *random*(*L*) and *Width* = *random*(*W*). This is a particularly good extension for Algebra students, because it involves generalizing functions for different parameters (see below).

Area and Perimeter, Extension 2



If $Length = random(L)$ and $Width = random(W)$, the results are similar to the results throughout this activity, except that many of the boundary functions are scaled accordingly. For the graph of *Area* versus *Perimeter,* part of the upper boundary is always $Area = \frac{Perimeter^2}{16}$ (and this is the only upper boundary if $L = W$). But if $L \neq W$, a second upper boundary comes into play: $Area = \left(\frac{\text{lesser of } L \text{ or } W}{2}\right) \cdot Perimeter -$ (lesser of L or $W)^2$. The lower boundary of *Area* versus *Perimeter* always depends on the greater of L or W:

$$Area = \frac{\text{greater of } L \text{ or } W}{2} \cdot Perimeter - (\text{greater of } L \text{ or } W)^2$$

Students might also notice that the upper boundary of the scatter plot of *Area* versus *Length* is always $Area = W \cdot Length$. For *Perimeter* versus *Length,* the boundaries are always $Perimeter = 2Length + 2W$ and $Perimeter = 2Length$.

Students can create sliders for L and W and confirm that their boundaries always work.

3. Statistics students can explore sample size by adding and deleting cases to see how the histograms and scatter plots compare for 10, 20, 50, 100, 200, 500, and 1000 cases. At what sample size can they begin to discern important patterns in the data? Are there pros and cons to using larger sample sizes? Students will probably find that they need at least 50 or 100 cases before they can describe patterns in the data. Even more cases, 500 or 1000, help "fill in" the patterns in the scatter plots, but don't help them see the pattern any better than smaller numbers.

4. For fun, show students how they can use the collection as a geometric representation of the rectangles. Drag the bottom corner of the collection to expand it. You'll see a gold ball for each rectangle. Double-click the collection to show the inspector and use the settings shown at right on the **Display** panel. The gold balls will change to rectangles.

If you asked the question "Why do you suppose so many of the points are close to the curved boundary?," students can now see that many of the rectangles are indeed close to being squares.