MAUNA LOA (PAGE 35)

Activity Time: 45–55 minutes

Required Document: MaunaLoa2003.ftm

Optional Documents: GlobalCarbon2000.ftm and **RegionCarbon2000.ftm** contain additional data that can be used with the Extension.

Fathom Prerequisites: Students should be able to

- Open a document

Fathom Skills: Students will learn how to

- Work with case tables
- Define an attribute with a formula
- Create graphs (scatter plots)
- Graph lines of fit
- *Optional:* Create a summary table and create a collection from its cells (Explore More 2)

Mathematics Prerequisites: Students should be able to make a scatter plot of points, write the equation of a line in slope-intercept form, use the slope of a line to describe rate of change, and use the equation or graph of a function to extrapolate values.

Mathematics Skills: Students will learn how to recognize patterns in a scatter plot of points, fit lines (and possibly curves) to model data, informally determine how well a function model fits a graph of data, and identify periodic behavior.

General Notes: In this activity, students focus on time series data and look at repeated measurements of atmospheric carbon dioxide (CO_2) over time. They'll look at trends in the data, namely that the concentration of CO_2 in the Earth's atmosphere is increasing. Although students have probably heard about the effects of global warming, in day-to-day life they may not be aware of atmospheric changes and could be skeptical of global-warming claims. This activity uses famous CO_2 measurements taken near the summit of Mauna Loa on the island of Hawai'i—the longest and best series of such data in the world. When students graph the Mauna Loa data, the increase in CO_2 is alarmingly obvious and concerns about global warming are more tangible.

You might begin with a brief discussion about global warming. Have students explain the basics of the problem: that greenhouse gases such as carbon dioxide (CO_2) and methane (CH_4) occur naturally in the atmosphere, but scientists believe that the concentrations are increasing as a result of human activity. It is further believed that this increase will cause global climate change as these gases trap infrared radiation (heat) inside the atmosphere. Ask, "If we are skeptical, what would we have to know to be convinced that this is true?" Briefly make a list of things students would want to know. Some connections are difficult to prove, of course, but there are also very simple questions: "Do we really emit a lot of CO_2?" and "Are the concentrations of these gases really increasing?" The latter question is the focus of this activity. The idea of human emissions can be one direction of research for the Extension.

While there are "official" techniques and representations commonly used in time-series data analysis, such as Fourier series and control charts, this activity uses an informal exploration. It is assumed that all Algebra 2 students are familiar with using linear equations to model real-world data. They have probably used informal methods (drawing a line that seems to fit) since middle school and may have already learned formal methods (median-median lines or least-squares lines). But students may have had experience only with "textbook" data that look almost perfectly linear when graphed. The data in this activity may surprise many students. While the general pattern is almost linear, it does have a slight curvature. And if you look closely at the seasonal changes between months, you'll notice a periodic wave in the data. (*Note:* In step 8, you should give your students explicit instructions about which lines of fit to use, depending on which lines of fit you have studied in class. If you have not studied any lines of fit, you may want to use step 8 as an opportunity to explore all three and discuss pros, cons, similarities, and differences.)

If you have used other activities in this book, you may notice that the document **MaunaLoa2003.ftm** does not use units. Even though Fathom does recognize the unit parts per million (ppm), units have been omitted here to make it easier to do Explore More 1. If ppm were used for the CO_2 concentrations, creating nonlinear functions would become complicated because you would have to include units in the function equations.

The conclusion of this activity—the last two questions, the Explore More ideas, and the Extension—is purposefully left open-ended. This gives all students an opportunity to problem-solve with data, deciding which data and graphs will help them answer different questions. It also makes the activity appropriate for Precalculus or Statistics students, allowing them to go beyond linear models and apply exponential functions, polynomial functions, trigonometric functions, or advanced methods of regression analysis. (See Explore More 1 for ideas about using nonlinear functions.)

MAKE A CONJECTURE

Q1 Students are probably aware that the amount of CO_2 in our atmosphere is increasing.

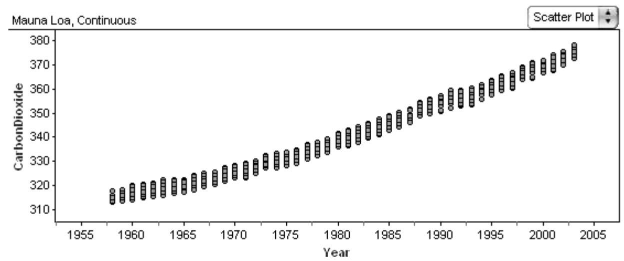
Q2 Some students might predict that the increase is linear. Others might predict a curve, such as an exponential function or a quadratic function.

INVESTIGATE

Q3 326.93 ppm. You find this from Mauna Loa, Continuous by scrolling down to find the value of attribute *CarbonDioxide* for *Year* 1970 and *Month* 3 (case number 147). You find this from Mauna Loa, By Month by finding the value of attribute *March* for *Year* 1970 (case number 13).

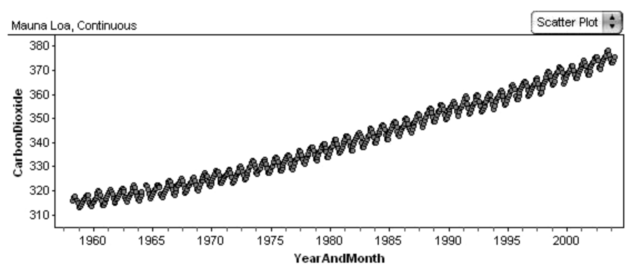
Q4 From looking at the case table for Mauna Loa, Continuous, students might have trouble noticing a pattern because the continuous values fluctuate. From the case table for Mauna Loa, By Month, students can clearly see that the concentrations steadily increase for any one month, but that they fluctuate for any one year.

Q5 Although the graph forms a wide band of points, the general pattern is increasing.



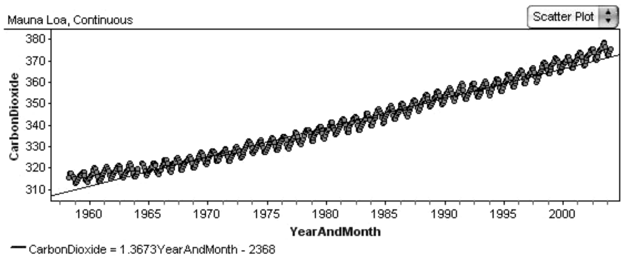
Q6 Students can use a variety of formulas for *YearAndMonth*. The simplest is $Year + \frac{Month}{12}$. Although this formula technically rounds December up to the next calendar year (for example, $1958 + \frac{12}{12} = 1959$), it is close enough for this informal analysis. You may want to ask students to share the different formulas that they used and to discuss the pros and cons of each.

Q7 Again, the general pattern is increasing. Students will likely mention the "wave" pattern formed by the data, which represents seasonal fluctuations in the CO_2 concentration.



Q8 Some students may not be able to see beyond the periodic nature of the data and will describe the shape as a "wave"; some of those students may describe the shape as "periodic" or "sinusoidal." Other students may look beyond the periodic nature and focus on the overall shape; they might describe the shape as "linear" or "slightly curved." In general, the shape is probably more complex than what students predicted in Q2.

Q9 Students will probably say that the line does not fit well. The median-median line shown in the next graph fits well in the middle portion of the graph, but the data on either end are a little too far above the line. This implies that a curve, such as an exponential function or a polynomial function, might be a better fit for the general pattern. Again, some students may not be able to see past the periodic nature, so they might recommend a trigonometric function of some sort.



Q10 The graph above shows an increase of about 1.37 ppm per year. (*Note:* The units are per year, not month, because *YearAndMonth* is a decimal part of a year.) Students' answers may vary anywhere from 1.0 to 1.5, especially if they use a movable line.

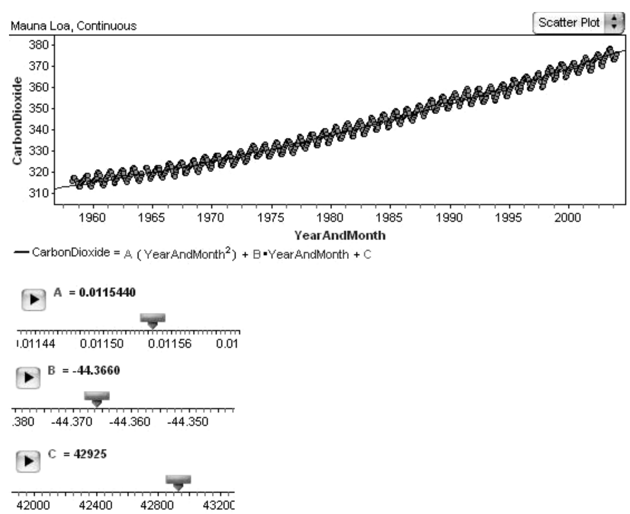
Q11 Techniques (and the current year) will vary. One technique is to trace the median-median line or least-squares line from the graph in step 8. This technique requires that students pick one value from any fractional part of the year; students might pick the beginning of the year (2005.0), the very middle (2005.5), or the end (2006). Another technique is to use Mauna Loa, By Month and create a new attribute that calculates the average concentration for each year, then make a scatter plot of the averages over time, fit a line, and extrapolate the average concentration for the current year. (Technically, this method is slightly flawed because the monthly concentrations themselves are averages of continuous readings. Mathematically speaking, the average of several averages is an invalid statistic.)

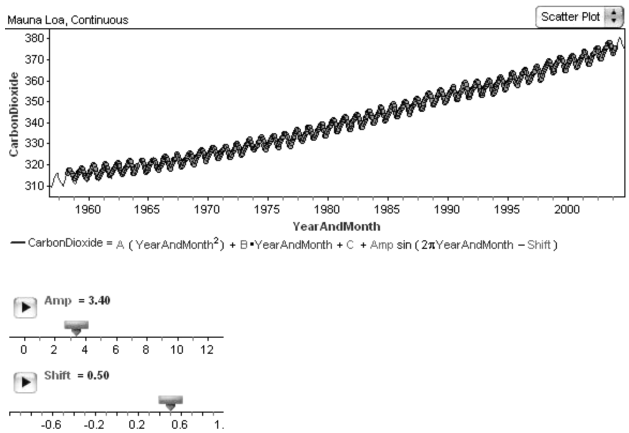
Q12 Techniques (and birthdays) will vary. One technique is to trace the line of fit for all of the data. In order to do this accurately, students will need to calculate their birth month (or, more accurately, birth day) as a decimal part of a year, trace to that point on the line, and then estimate the amount of fluctuation above or below the line for that month (or day). Another technique is to use Mauna Loa, By Month and use a scatter plot of only the student's birth month. This technique is a nice way to determine the average CO_2 concentration for any given month, but it does not lend itself to predicting any particular day.

EXPLORE MORE

1. This question extends the activity into nonlinear models, specifically exponential, polynomial, or trigonometric. By combining the general curvature of the data and the periodic seasonal fluctuation, students could even be challenged to write and graph the sum of two functions. Fathom does not calculate nonlinear models, so students will need to use trial and error or transform the data by creating new attributes.

   By trial and error, students may find that a quadratic function best fits the general curvature of the data. Sliders can be used for the parameters to narrow in on an equation similar to $\mathit{CarbonDioxide} = 0.011544(\mathit{YearAndMonth})^2 - 44.366\mathit{YearAndMonth} + 42{,}925$. Then, looking at the amplitude, period, and phase shift of the wave, and using sliders, students may be able to append the function with $+\ 3.4\sin(2\pi\mathit{YearAndMonth} - 0.5)$.





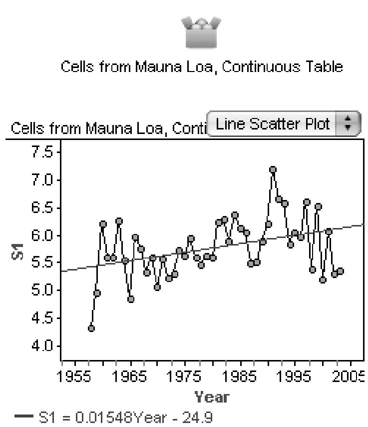
2. This might be a good time to reinforce (or introduce) terminology related to periodic behavior: *maximum* (crest), *minimum* (trough), *amplitude, period,* and so on. With very few exceptions, the maximum crests occur in May and the minimum troughs occur in September or October.

   An interesting way to analyze the range of fluctuations is to create a summary table for Mauna Loa, Continuous that calculates the range of carbon dioxide concentrations for each year. You can then create a collection from the cells in the summary table (see Fathom Help: Create a Collection from the Cells of a Summary Table) and make a graph of the ranges over time. There is a great amount of variation in the ranges, but there is a slight, overall increasing pattern.

Mauna Loa, Continuous

1958	4.31
1959	4.95
1960	6.2
1961	5.59
1962	5.59
1963	6.25
1964	5.53
1965	4.86
1966	5.97
1967	5.74
1968	5.32
Column Summary	65.16

S1 = max(CarbonDioxide) − min(CarbonDioxide)



EXTENSION

Have students do additional research on carbon dioxide, other greenhouse gases, and global warming. Their research might take them in a variety of directions. Some students might want to do an essay or a paper on exactly what global warming is, why it is a concern, and what scientists and environmental agencies are doing to counteract it. Others might research scientists' current analysis of greenhouse gases and global warming and describe how their Fathom analysis of the Mauna Loa data is consistent with formal studies. Still others might find additional data on CO_2 (such as the amount of CO_2 that human activity contributes to the environment), analyze it, and explain how it fits with the Mauna Loa data. Or students might find data on other greenhouses gases, such as methane (CH_4). A good place to begin researching is the Carbon Dioxide Information Analysis Center (CDIAC), which you can find on the Internet at cdiac.esd.ornl.gov.

GlobalCarbon2000.ftm and **RegionCarbon2000.ftm** contain additional data gathered from the CDIAC Web site. These documents give carbon dioxide emissions—the amount humans put into the atmosphere, as opposed to the amount that is naturally part of the environment—from various sources around the world. Each file gives the total emissions in million metric tons of carbon and breaks down the total into component parts: *Gas* (e.g., natural gas), *Liquid* (e.g., petroleum), *Solid* (e.g., coal), *Cement* (carbon dioxide from cement production), and *Flaring* (the burning of gas at refineries and drilling sites). The documents also give *PerCapita* carbon dioxide production in metric tons per person per year. **GlobalCarbon2000.ftm** gives measurements for the whole world, while **RegionCarbon2000.ftm** breaks down the emissions by geographic regions.

FUNCTION TRANSFORMATIONS (PAGE 38)

(PAGE 38)

Activity Time: 50–100 minutes

Required Document: FunctionTransformations.ftm

Optional Document: FuncTransformPRE.ftm

Fathom Prerequisites: Students should be able to

- Create attributes defined by formulas
- Make a scatter plot of two attributes

Fathom Skills: Students will learn how to

- Create sliders
- Plot functions
- Duplicate graphs
- Link graphs' axes

Mathematics Prerequisites: Students should be able to graph linear, quadratic, absolute value, exponential, square root, and inverse-variation functions; and identify the general shapes of linear, quadratic, absolute value, exponential, square root, and inverse-variation functions. (*Note:* If students know only a few of these functions, you can modify the activity so that they work only with the functions they know.)

Mathematics Skills: Students will learn how to transform data, transform functions, identify the transformation that results from different function parameters, and determine the function that fits data based on transformations.

General Notes: This activity is different from most Fathom activities. Instead of analyzing real-world data that have reasonable amounts of variability, this activity uses data generated by functions. This allows you to use Fathom as a tool for algebraic manipulation.

In this activity, students will transform a set of quadratic data (and possibly data generated by other functions) and see how the equation of the curve that fits the data changes. Students will learn how the parameters h and k (translations) and a and b (dilations) affect the graph of a function. This gives students in-depth experience with multiple representations of function: table, graph, and algebraic equation.

If you did the activity Slope-Intercept Form on pages 11–14, the format of this activity will be familiar. Or, if you make extensive use of graphing calculators in your algebra class, the structure of this activity may also seem familiar. Transforming the data by creating new attributes is very similar to creating lists on a graphing calculator.

Be aware that there is a subtle distinction between transforming data and transforming a function that might confuse some students. For example, consider that you have data in the form (x, y) from the parabola $y = x^2$. When you transform the data into the form $(x, y + 2)$ by translating each point 2 units up, the parabola that fits is now $y = x^2 + 2$. Some students may be puzzled because you add to the y-coordinate, yet the equation compensates by adding to the x-side of the equation. One explanation is that the equation must maintain the same equality as before the data were transformed, or $y = x^2$. So, if you add to the y-coordinate, the equation must add the same amount from the x-side—substituting $(x, y + 2)$ into $y = x^2 + 2$ gives $y + 2 = x^2 + 2$, or $y = x^2$. Another explanation is that the equation must undo the transformation in order to be consistent with the original data. For this explanation, rewrite the transformed equation as $y - 2 = x^2$; when you substitute $(x, y + 2)$, you get $y + 2 - 2 = x^2$, or $y = x^2$. The latter explanation is sounder because it helps explain a wider array of function transformations. In general, it explains why data in the form (x, y) from the function $y = f(x)$ can be transformed into $(x + h, y + k)$ and the function $y - k = f(x - h)$.

This activity encourages students to explore several function transformations (translations, reflections, and dilations) and several different functions (linear, quadratic, absolute value, exponential, square root, and inverse variation). To do all of the possible combinations will take seasoned Fathom users a long time; it will take novice Fathom users even longer to become comfortable with making attributes, sliders, and graphs. Hence, you should plan at least two days to complete the entire activity. One option is to have students explore one transformation for all functions on each day. That is, explore translations for all functions, and then explore dilations for all functions. If you want to do reflections or rotations, you can squeeze them in during the second day or use a third day of exploration. (The worksheet is already organized by different transformations to help you follow this option.) Another option is to explore all transformations for one function on each day. That is, explore translations and dilations (and possibly reflections and rotations) for just quadratic functions; then explore all transformations for just absolute value functions; and so on. This option is particularly useful if your students haven't learned about all of the different functions yet. You could even revisit the activity throughout the school year as you introduce each new function.

Part of the goal of this activity is to teach students how to use several of Fathom's features, such as sliders and duplicate graphs. If your students are already familiar with Fathom, or you want to save the time necessary to teach the software to novice students, you may prefer to use **FuncTransformPRE.ftm.** This "pre-made" document has the sliders and graphs prepared for you and your students. That way, students can complete the worksheet by answering only the questions beginning with Q1.

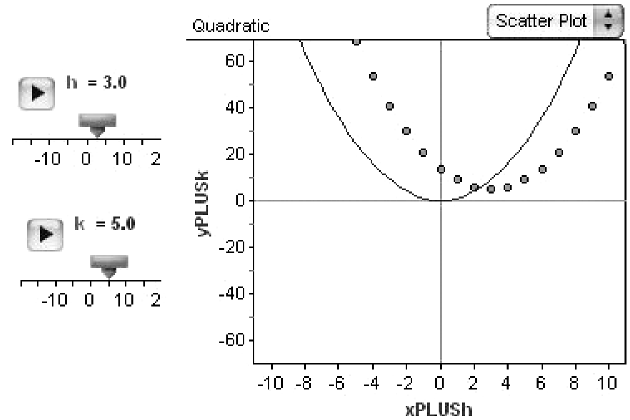
Understanding function transformations is frequently taught in Algebra 2, but a few curricula teach them or review them in Precalculus; and a few curricula introduce function transformations in Algebra 1. Therefore, depending on your curriculum, this activity could be successfully used in higher- or lower-level courses.

INVESTIGATE

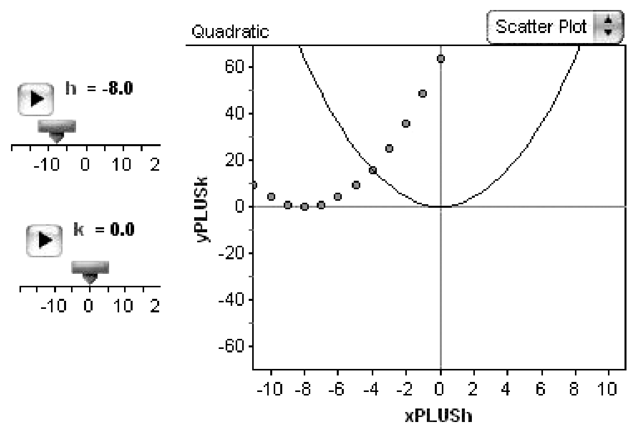
Q1 Changing h makes the points move left (negative) or right (positive) by that number of units.

Q2 Changing k makes the points move down (negative) or up (positive) by that number of units.

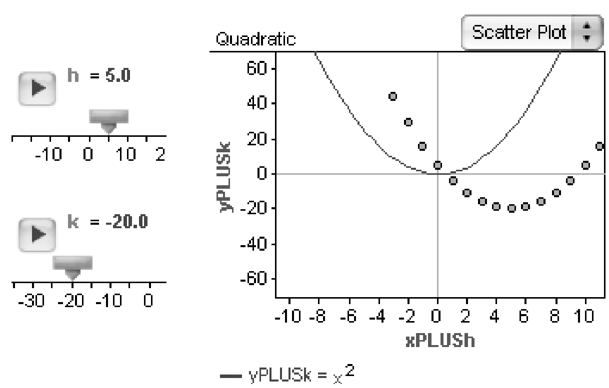
Q3 a. The points will be moved right 3 units and up 5 units.



b. The points will be moved left 8 units.

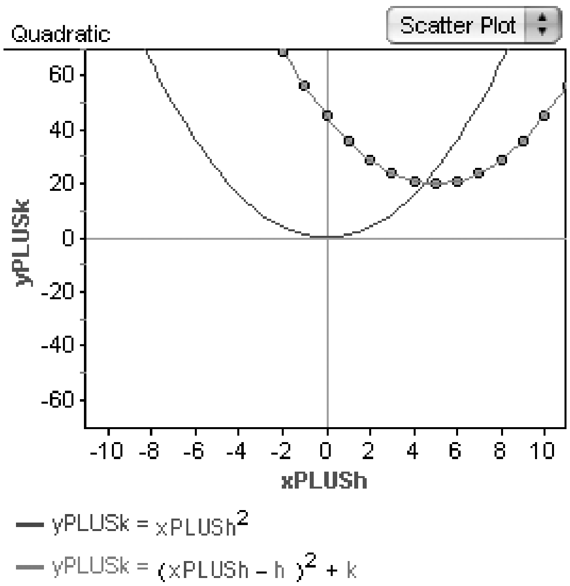


c. The points will be moved right 5 units and down 20 units.



Q4 The coordinates of the vertex are (h, k).

Q5 $yPLUSk = (xPLUSh - h)^2 + k$



Q6 Students should find that the same translation rules apply to all functions. In general, h and k will transform $y = f(x)$ into $y = f(x - h) + k$, or $y - k = f(x - h)$.

Hint: Ask students to explain what point(s) they consider to be like the vertex of a parabola when answering Q4 for the other functions. For absolute value and square root functions, students may recognize an analogous vertex. However, the other functions require some ingenuity.

Note: To explore the other collections, students need to create case tables for the data in a collection. The simplest method is to select the collection and then drag a case table from the shelf.

Q7 At a minimum, students should mention that a shrinks or stretches the graph horizontally and b shrinks or stretches it vertically. Some students may give more detailed descriptions, describing what happens for values greater than 1, equal to 1, between 0 and 1, equal to 0, between 0 and -1, equal to -1, and less than -1.

Q8 $bTIMESy = b\left(\frac{xTIMESa}{a}\right)^2$

Q9 Answers will vary, but students should find that the same dilation rules apply to all functions. In general, a and b will transform $y = f(x)$ into $y = b \cdot f\left(\frac{x}{a}\right)$, or $\frac{y}{b} = f\left(\frac{x}{a}\right)$.

Q10 When $a = -1$, there is a reflection across the y-axis. When $b = -1$, there is a reflection across the x-axis.

Note: Because both quadratic and absolute value functions have the y-axis as a line of symmetry, suggest that students use other functions when exploring the effect of $a = -1$. They should then be able to explain why the transformation has no effect on those functions.

EXPLORE MORE

1. Students should find that dilations and translations work together the same way they do individually. In general, a, b, h, and k will transform $y = f(x)$ into $y = b \cdot f\left(\frac{x-h}{a}\right) + k$, or $\frac{y-k}{b} = f\left(\frac{x-h}{a}\right)$. The only thing that may challenge some students is realizing that a and b are the quotients for the entire translations.
2. $(-x, -y)$ creates a 180° rotation about the origin; $(-y, x)$ creates a 90° counterclockwise rotation about the origin; and $(y, -x)$ creates a 90° clockwise rotation about the origin. Students may find that rotations are different in at least two ways. First, some rotations require that the coordinates be switched. Second, the rotations that require switching coordinates are not always functions; under translations, dilations, and reflections, the image of a function is always a function.

MOORE'S LAW (PAGE 42)

Activity Time: 45–55 minutes

Required Document: MooresLaw2004.ftm

Fathom Prerequisites: Students should be able to

- Make a scatter plot of two attributes
- Plot functions
- Create sliders
- *Optional:* Create attributes defined by formulas

Fathom Skills: Students will learn how to

- Trace a function to extrapolate
- Make residual plots

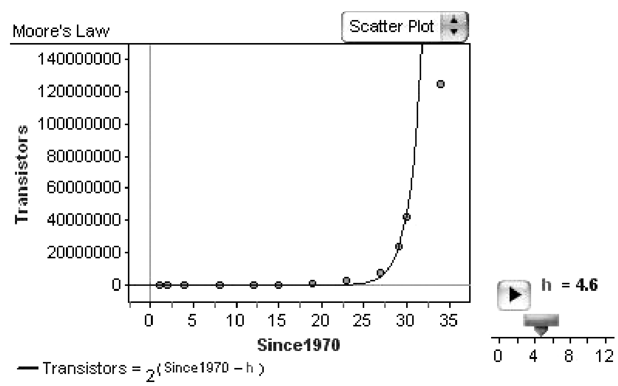
Mathematics Prerequisites: Students should be able to graph exponential functions and transform functions.

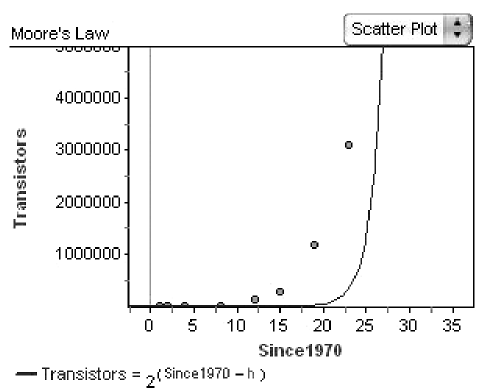
Mathematics Skills: Students will learn how to fit an exponential function to real-world data by trial and error, interpret doubling time from an exponential function's parameters, and use residuals to judge how well a function fits data.

General Notes: This activity explores data about the exponential growth of the number of transistors that can be placed on a computer's central processing unit. The growth was originally observed by Gordon Moore in an article titled "Cramming More Components onto Integrated Circuits" (*Electronics,* vol. 38, no. 8, April 19, 1965). In his famous article, Moore predicted that the "number of components per integrated function" (frequently interpreted to mean "number of transistors") would double about every year. In subsequent years, Moore himself and others have revised the prediction to be a doubling about every 1.5 to 2 years. This activity gives students an opportunity to find their own function of best fit and either agree or disagree with Moore's prediction.

This activity is intended to closely follow the previous activity, Function Transformations (pages 38–41). Here students use sliders and transformations to adjust an exponential function in the form $f(x) = b \cdot 2^{(x-h)/a} + k$ until it fits the data. If you have not done Function Transformations, you may want to do it first. Otherwise, plan to give more detailed instructions for step 4 of the student worksheet.

In step 4, adjusting the exponential curve with sliders to get the "best" fit is an extremely open-ended and subjective process—there is no right answer. Encourage students to try different combinations of transformations. The process is also complicated by the range and vertical scale of the graph, which makes some points appear to fit the function when they are actually quite removed from it. Therefore, encourage students to zoom in to small portions of the graph, especially the first eight points, to see how well the function really fits. For example, if students try only a horizontal translation, it may look like the function fits eleven of the twelve points well. However, zooming in to the first eight points shows that the curvature isn't quite right. In order to fix the curvature, students need to incorporate dilations as well.





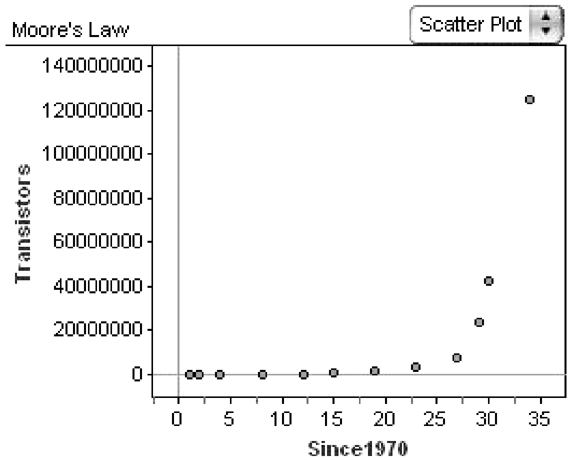
After students find a curve that fits, they need to be able to identify which parameter in the function represents doubling time. Pinpointing the doubling time allows students to compare their model to Moore's original prediction of doubling once every year. If your students have not studied doubling time or half-life, you can guide them toward the correct parameter by asking, "Which variable in the equation represents time?" and "What value of time would you need to get 2^1? 2^2? 2^3?" and "How many years are there between each multiple of 2?"

After finding a function of fit by trial and error with sliders, the activity introduces students to residual plots as a way to judge and refine the fit. This introduction is intended to be informal, giving Algebra 2 students the opportunity to explore how the residuals change as the function changes and helping them learn to identify points that fit relatively well versus those that don't. Again, the range and scale of the residual plot makes it look like the first five or six points always fit with a residual of zero, when actually they may not. If you use this activity with a Precalculus or Statistics class, you may want to encourage students to zoom in to the residuals of those points to get a clearer picture. Students in these classes might also benefit from showing the residual squares and trying to minimize the sum of the squares.

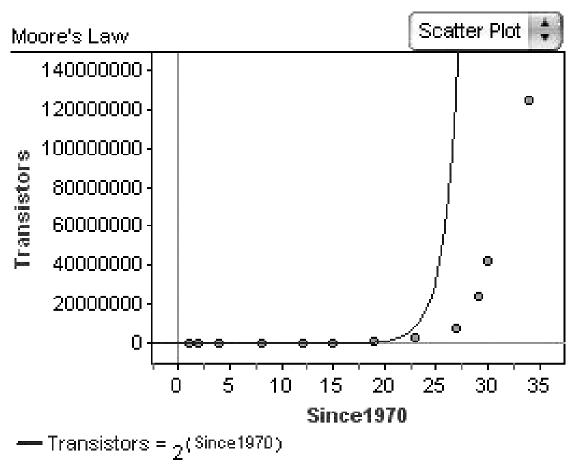
Precalculus or Statistics students can also benefit by doing a semi-log transformation of the data. Taking the logarithm of *Transistors* nicely linearizes the data. Students can use a least-squares line to extrapolate, or they can use powers and logarithms to rewrite the linear equation as an exponential function (with base 10 or base 2) that fits the original data. (See Explore More 2 for additional details.)

INVESTIGATE

Q1 Students may use a variety of verbal descriptions, hopefully including the word *exponential.* The graph implies that the rate at which the number of transistors is growing is increasing over time.

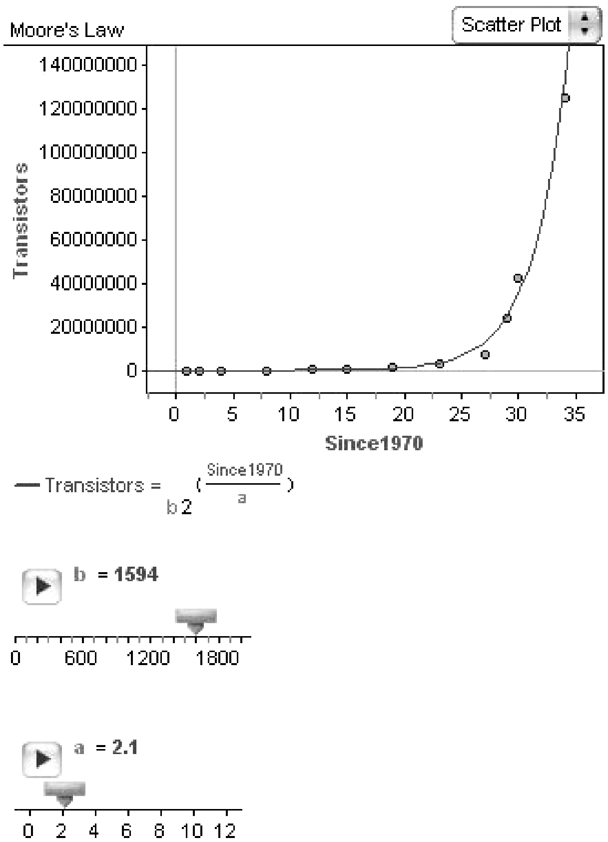


Q2 The function appears to fit the first seven points but definitely doesn't fit the last five points. If students zoom in to the first seven points, they will see that these points don't fit very well either. Students might think that a horizontal translation would improve the fit.



Q3 Some students may try only a horizontal translation. However, as explained in the general notes above, a translation misses the twelfth point and doesn't really fit the other points. Other students may find that a combination of dilations helps fit the data the best. Yet other students may try combining dilations and translations. Because the curve's upward bend is in the same direction as the points' upward bend, students will probably not try reflections.

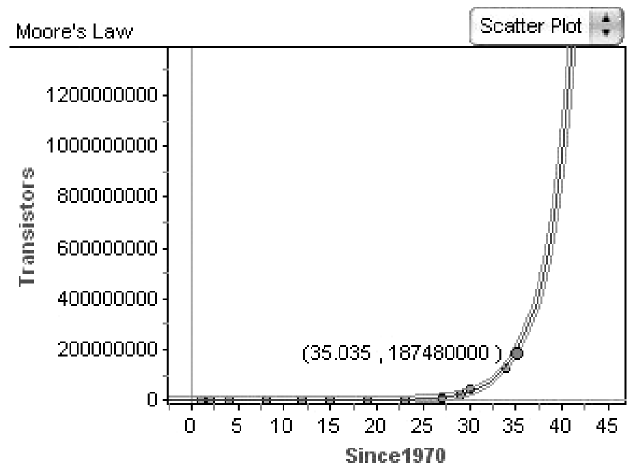
Q4 The graph below shows the function *Transistors* = $1594 \cdot 2^{(Since1970/2.1)}$, which is the result of two dilations.

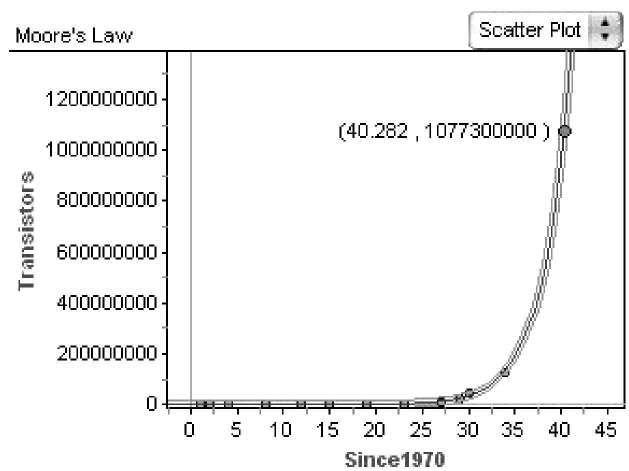


Q5 Students should pick out the divisor of the exponent, or the parameter *a*, as the doubling time. For the example in Q4, the doubling time is 2.1 years. For students who use only a translation, the value of *a* is 1.

Q6 The exact answers the student gets depend on the method used to extrapolate. Direct substitution gives approximately 166 million transistors in 2005 (or 35 years since 1970) and approximately 864 million transistors in 2010 (or 40 years since 1970). If students trace the graph in Fathom, then the answers depend on the resolution used; the graph is not truly continuous, so not all values will show up. For the example in Q4, tracing gives approximately 188 million transistors for *Since1970* = 35 and over 1 billion transistors for *Since1970* = 40.

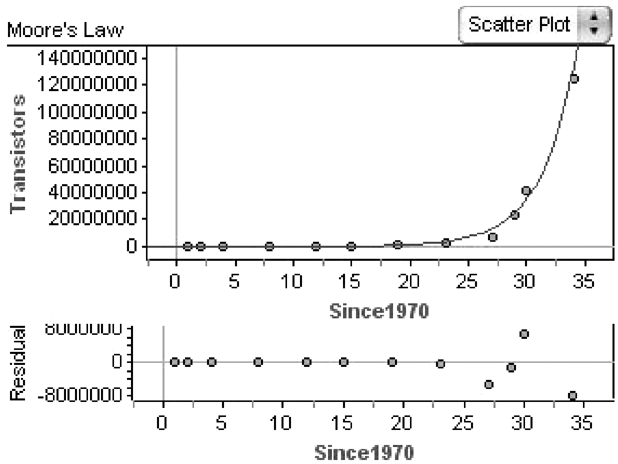
Note: Students may want to change the domain and range of the graph's axes so that they can adequately see the traced points.





Q7 If the function passes through the data point, the residual point will be on the gray zero line. If the function doesn't pass through the data point, the residual point will be above or below the zero line. Students should notice that the residual grows larger as the function's fit worsens.

Q8 The points for the years 1997, 1999, 2000, and 2004 are the most problematic. As you adjust the function with the sliders, these residual points fluctuate wildly and it is very hard to make the function fit all of them well.



Q9 See the answer to Q4 for one possible function.

Q10 Students will probably agree that "complexity" varies exponentially. Doubling times and functions will vary.

Q11 Some students may be persuaded by the data in this activity and feel that if the number of transistors has grown exponentially for the last 35 years, then the trend is likely to continue. Other students may argue that the growth might reach a maximum and eventually taper off (similar to a logistic function). According to Intel Corporation's Web site, the company believes the trend will continue through at least 2010.

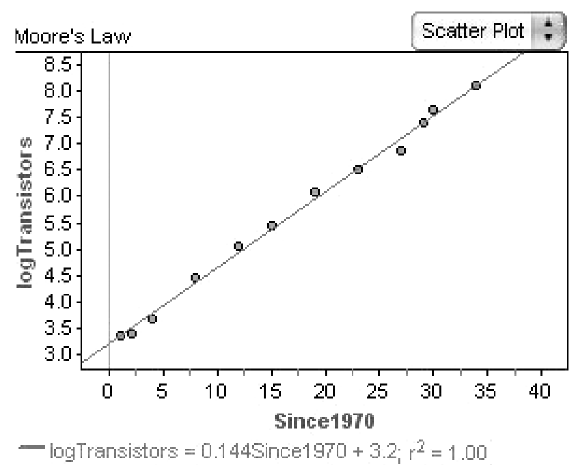
EXPLORE MORE

1. Suggest that students research Intel's Web site, www.intel.com/research/silicon/mooreslaw.htm, which was the original source for the data presented in the Fathom document **MooresLaw2004.ftm.** In September 2004 (after the Fathom document was compiled), Intel published data for the Itanium line of processors. The Itanium was released in 2002 with 220,000,000 transistors, and the Itanium 2 was released in 2003 with 410,000,000. Hence, these processors far exceed the exponential trend of the data used in this activity.

   Students can add the Itanium transistors, and any others that they research, to the Fathom document as new cases. They can then readjust the exponential function to obtain a better fit. Some students may find that no single function fits both the older and newer processors, so they might try to fit two or more exponential functions to create a piecewise-defined function.

2. Creating a semi-log graph of *logTransistors* and *Since1970* linearizes the data. Students can use a line of fit (least-squares line shown below) to model the data. The least-squares line can be traced to extrapolate, or students can convert to an exponential function. Using x and y, here's a conversion from the linear function to the exponential function:

$$\log(y) = 0.144x + 3.2$$
$$10^{\log(y)} = 10^{0.144x+3.2}$$
$$y = 10^{0.144x} \cdot 10^{3.2}$$
$$y = 2^{0.144/\log(2)} \cdot 1585$$
$$y = 1585 \cdot 2^{0.478x}$$
$$y = 1585 \cdot 2^{(x/2.09)}$$



EXTENSION

As people living in the age of technology, students may take it for granted that computers are going to become more powerful every few years. Because of this, they may not realize how impressive the growth of computer technology is. To get a better perspective, have students do research on another type of technology, machinery, or industry and research its maximum performance or minimum cost in 1970. For example, they could research the maximum screen size of a television in 1970; or the price of a transcontinental flight in 1970; or another similar situation that would involve growth or decay. Apply Moore's Law (growth or decay version) to that technology, machinery, or industry. According to Moore's Law, what should the value be today? Then research the actual values today and compare.

At the time of this writing, Intel Corporation's Web site has a nice (if not lengthy) animated presentation about Moore's Law that includes an analogy to air travel. Visit www.intel.com/labs/eml/eml_demo/EML_demo.htm and jump to "An Analogy" to see this analogy. If you don't want to assign this extension as a short research project, you might be able to use this Internet presentation as a classroom demonstration instead.

After doing research, or after watching the Internet presentation, have a class discussion about whether students think it is legitimate to apply Moore's Law to other forms of technology, machinery, or industry. Some may argue that Moore's Law might apply to other areas if you didn't look at just 1970 to the present. For example, between the Wright Brothers' first flight (1903) to Howard Hughes' first worldwide flight (1938), the field of aviation developed at an impressive rate, too. Other students might argue that the speed at which computer technology develops does create pitfalls—for example, computer crashes and the consumer's cost of continually upgrading—that would be unacceptable in other products.

PRINTING PARAGRAPHS (PAGE 45)

Activity Time: 45–55 minutes

Materials:

- One copy of The Seven Paragraphs worksheet per group (page 49)
- Centimeter rulers

Optional Document: PrintingParagraphs.ftm

Fathom Prerequisites: Students should be able to

- Create a collection by starting a case table
- Make a scatter plot of two attributes
- Plot functions
- Make sliders

Fathom Skills: Students will learn how to

- Make a residual plot
- Use a residual plot to improve a function model

Mathematics Prerequisites: Students should be able to graph inverse-variation functions. *Optional:* Students should be able to use function transformations (dilations and translations) and fit an exponential function by trial and error.

Mathematics Skills: Students will learn how to fit an inverse-variation function to real-world data by trial and error, use residuals to judge how well a function fits data, and use residuals to improve the fit of a function.

General Notes: This activity has two main goals. First, it teaches a real-world application of inverse-variation functions. Specifically, it looks at the inverse relationship between paragraph width (horizontal) and length (vertical) assuming that the amount of text, or the area, remains the same. Second, it introduces residual plots (or re-introduces, if you did Moore's Law) as a means of judging how well a function fits data and how you might be able to adjust the function model to improve the fit.

For the first goal, students need to first recognize that the data follow an inverse variation. Even very experienced students, when seeing how *Length* decreases with *Width* to form a curve, will say, "It must be exponential." To overcome this misconception, the worksheet encourages students to think about the geometric relationship ($Length \cdot Width =$ Area) and then directs them to the function $Length = \frac{Area}{Width}$. By using a slider for the parameter *Area,* students can use the same curve-fitting methods that they used in Function Transformations and Moore's Law.

You may want to formally introduce the term *parameter* and fully distinguish it from a *variable* (or *attribute,* in Fathom's lingo). Although students will use a slider to *vary* the parameter *Area,* it is not a *variable.* Technically they are trying to find the best value of a *constant,* so *Area* is a *parameter.*

Please be aware that some students, especially English-as-a-second-language students, may get confused by the multiple uses of *length* in this activity. In the context of the activity, the length of the paragraph (the attribute *Length*) is how long the paragraph is when measured vertically. However, the attribute *Width* is also measuring a length—line length. For this reason, the worksheet avoids the phrase *line length,* even though it is a widely used term in the publishing industry. Despite any potential confusion, using *Length* and *Width* will help students recognize the connection to the area formula for rectangles, which is fundamental to the activity.

For the second goal, this activity intends to informally introduce residual plots. Residual plots show how far the data points in a scatter plot are from some function. By adjusting the slider(s) in this activity, students should get a feel for what a residual plot looks like for a good fit (most residuals close to zero) and what it looks like for a bad fit; what it means if all of the residuals are positive versus negative; and possibly what it means if you recognize a pattern in the residuals. Looking at residuals can tell you at least two things: whether your model is missing something and how accurately your model predicts the data. (The former concept is tackled when students add *White* to their function in Explore More 1.) In addition, just playing with Fathom's dynamic residual plots will give Algebra students more intuition about functions and how they're put together.

You may want to explain to students that there are two fundamental sources for residuals: errors and mistakes. Without getting too deeply into formal statistical definitions, the difference is that errors are unavoidable. In data analysis, you want to minimize and measure error and to avoid mistakes. In the context of this activity, measuring the paragraphs to the nearest tenth of a centimeter means that you will have unavoidable variability from the exact measurement—that's an error. On the contrary, modeling your data with a function that doesn't consider the ragged right margins means that your model is inherently wrong—that's a mistake. Sometimes it is hard (or even pointless) to tell the difference between an error and a mistake. But more often, distinguishing between these categories and their causes will help you make better sense of your data. For this activity, recognizing the mistake (omitting the ragged right) allows you to improve the function and find a better fit.

Another source of variability in this activity is how students measure the width of the paragraphs. Some students may measure the longest line; some may measure the shortest line; and some may try to approximate a median width for each paragraph. If you prescribe that certain groups use a particular method, you can pool the entire class's data into a single Fathom document and see how the data points and function models compare.

If time is a factor, you may prefer to have students use the pre-made document **PrintingParagraphs.ftm** instead of making their own measurements. This document already has measurements for *Width* and *Length* entered into a case table. By using this document, you can have students skip the Experiment section of the worksheet.

If you plan to use this activity with Precalculus or Statistics students, do not overlook Extension 2, in which students are challenged to find ways to straighten the data, find a linear model for the transformed data, and then convert the linear equation back into a nonlinear function for the original data.

MAKE A CONJECTURE

Q1 Students might say the graph will "decrease" and/or that it will "curve." Some students who visualize a curve might describe it as "exponential" or "decay." Depending on how recently you studied this type of function, a few students might say that it will be an "inverse variation."

Q2 Each paragraph is approximately a rectangle.

Q3 $Length \cdot Width = Area$, or $Length = \frac{Area}{Width}$

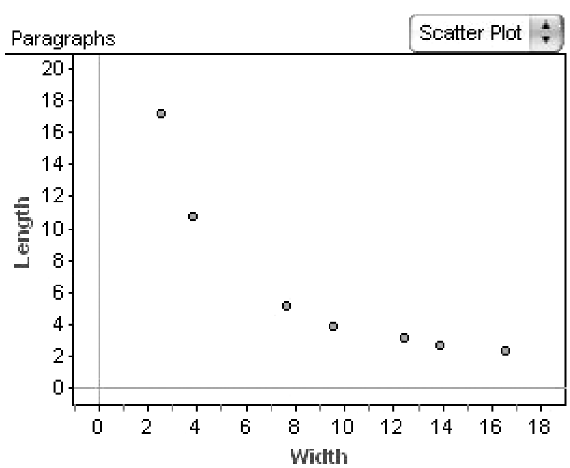
Q4 If students think of the area as consisting of only the text itself, they will say "yes" because each paragraph contains the same text. Some students may think of the area as the area of the rectangle that fits around the text, so they will say "no" because some rectangles have more white space in the right margin than others.

EXPERIMENT

Q5 Answers will vary, especially for measuring width. Some students will measure width with the longest line; some with the shortest line; and some with a median ("average") line.

INVESTIGATE

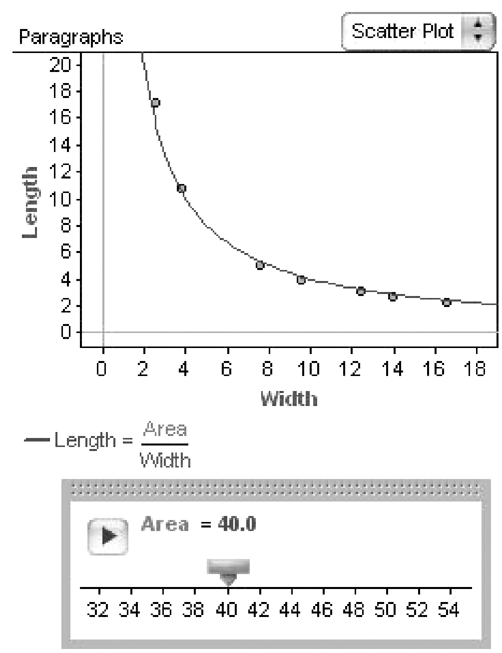
Q6 Here's a sample graph based on measuring the longest line for *Width*. Answers to "How does it compare?" will vary.



Q7 Hopefully some students will notice that smaller values of *Area* are required to fit the right-most points (the paragraphs with greater *Width*) and larger values of *Area* are required to fit the left-most points (the paragraphs with smaller *Width*). In terms of the printed paragraphs, this pattern exists because the narrower paragraphs have more words that wrap to the next line, so they end up being disproportionately longer and taking up a larger area.

Q8 No, the paragraphs do not have exactly the same area because no one value of *Area* fits all of the points. Yes, the areas are approximately the same because it is easy to make the curve fit very closely (not exactly) to each point.

Q9 Values around 40 are reasonable. Explanations might include that most of the points are close to the curve or that if you multiply any pair of *Length* and *Width* measurements you'll get a value around 40.



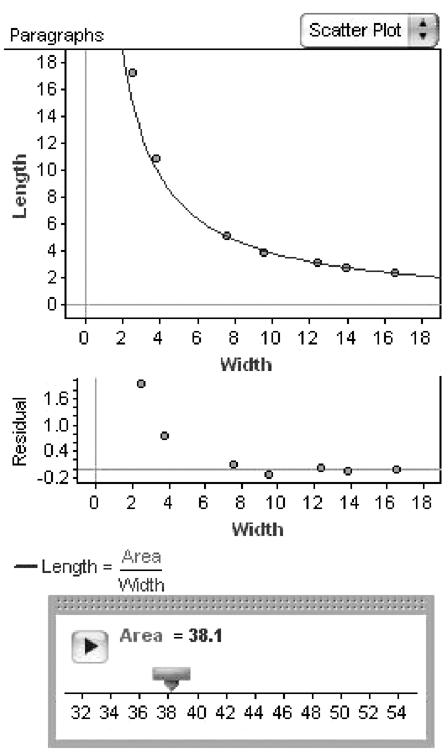
Q10 Ranges between 37 and 41 are reasonable.

Q11 The values of A show that the paragraphs' areas do increase as their widths decrease. They also show a range of 37 to 43.

	Width	Length	A
1	16.5	2.3	37.95
2	13.9	2.7	37.53
3	12.4	3.1	38.44
4	9.5	3.9	37.05
5	7.6	5.1	38.76
6	3.8	10.8	41.04
7	2.5	17.2	43

Q12 Students will probably find that the best value is somewhere in the middle of the range of A. However, it probably isn't equivalent to the median or mean of the values.

Q13 A value around 38 is reasonable because, although it leaves the first two points high, the majority of points are very close to zero. Students may say that if the majority of points are on one side of the curve, it can't be right. Others may reason that the two narrowest paragraphs (the left-hand points) look funny anyway, so you consider them outliers and make residuals as close to zero for the other five.

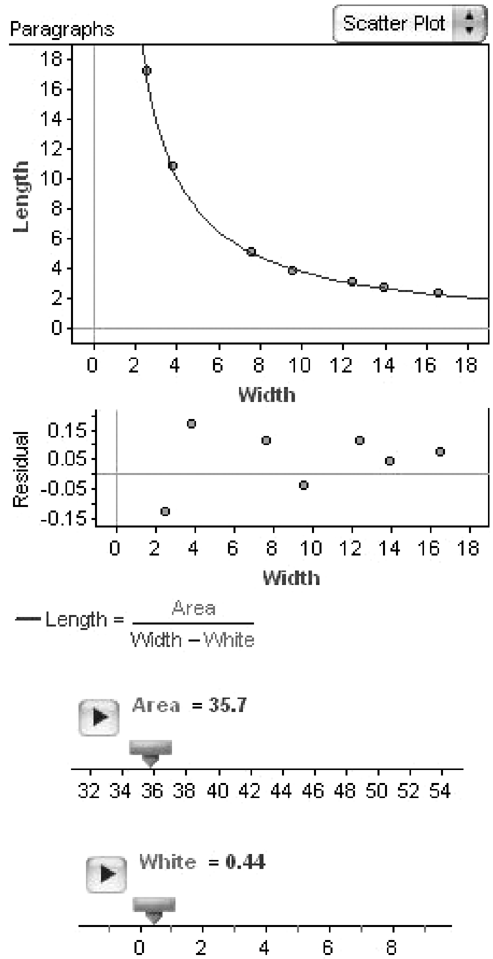


Q14 When focusing on the residual plot, many students will have a smaller value of *Area* than before.

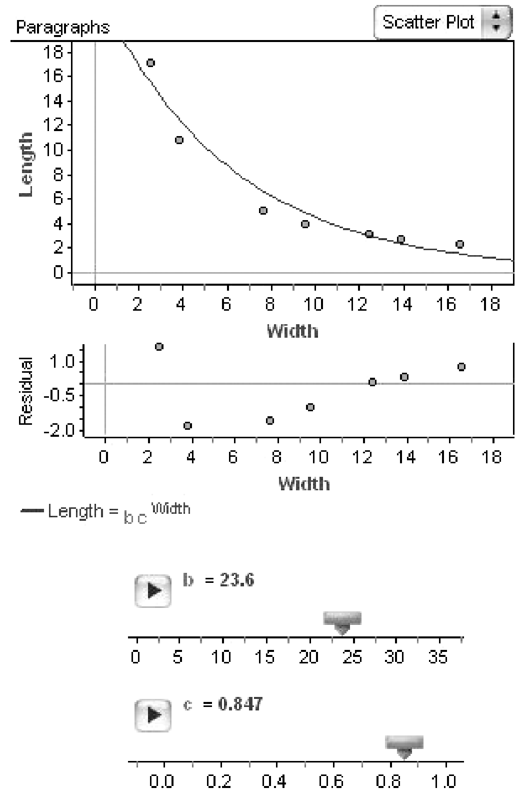
Q15 Because the worksheet introduces (or reviews) the term *inverse variation*, many students will try to use that to describe the relationship between *Length* and *Width*. Many will also mention that *Area* is a key factor. However, based on the fact that *Area* increases as *Width* decreases or based on the imperfect residual plot, most students will say that the model is not the best possible. Some students may realize that the white space at the end of each line needs to be considered in order to improve the model.

EXPLORE MORE

1. As shown in the following sample graph, using the parameter *White* significantly reduces the residuals. The residuals for the two left-most points are impacted the most. These points represent the two paragraphs with the smallest *Width* and also the paragraphs in which words more frequently had to wrap to the next line. So, it makes sense that white space has more impact on these paragraphs. Students should also find that the value for *Area* is smaller after incorporating *White*. The reason is that by subtracting the white space from the right edge, you decrease the overall width of the paragraph. So, the area also needs to decrease in order to maintain the proportion. *Note:* Students who measured *Width* based on the smallest line should have a negative value for *White*. Students who used an average line should have a very small value for *White*.



2. Mathematically speaking, *Area* dilates the curve toward the origin (both horizontally and vertically) and *White* translates the curve horizontally. For the residual plot, *Area* dilates toward the zero line with about proportionate changes in each residual and *White* also dilates but with a greater influence on the first two points (the outliers).
3. The geometric relationship is not exponential, so obviously students will have trouble fitting an exponential curve as well as the inverse-variation curve. Notably, students should find that the curvature is never quite right. The graph below shows one possible fit.



EXTENSIONS

1. Why would anyone ever need to predict the length of a paragraph? Hold a class discussion or have students do research on professional careers that might be concerned about the size of printed paragraphs. People who set type care about lengths of text—for example, newspaper layout people, advertisers, book publishers, magazine editors, and so on. Other aspects routinely considered in those fields are font size, general page layout (including artwork and photos), and readability.
2. Have students try transforming the data to get a linear pattern. Specifically, create an attribute *recipLength* (the reciprocal of *Length*) and graph *recipLength* versus *Width*. Talk about why this data transformation creates a linear pattern. Add a least-squares line to the graph and challenge students to algebraically transform the linear function into a

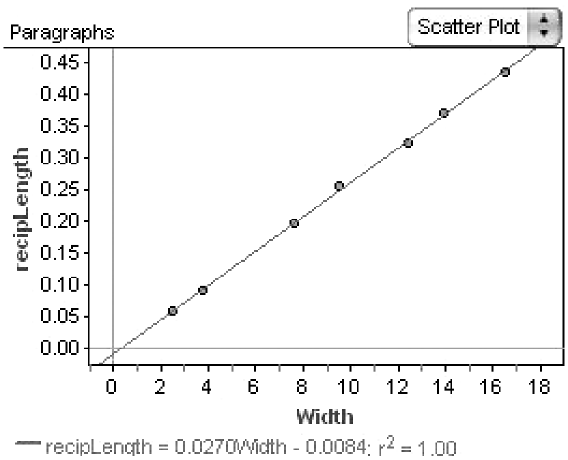
nonlinear function for the original data. For example, the linear function in the graph below could be transformed as follows:

$$\frac{1}{Length} = 0.0270\,Width - 0.0084$$

$$\frac{1}{Length} = 0.0270(Width - 0.3111)$$

$$Length = \frac{1}{0.0270(Width - 0.3111)}$$

$$Length = \frac{37.0370}{(Width - 0.3111)}$$



HOW MUCH PAPER IS LEFT? (PAGE 50)

Activity Time: 45–55 minutes

Materials:

- One new roll of adding-machine paper for each group
- Millimeter rulers
- Metersticks

Optional Document: HowMuchPaper.ftm

Fathom Prerequisites: Students should be able to

- Create a collection by starting a case table
- Work with units
- Create attributes defined by a formula
- Make a scatter plot of two attributes
- Graph a least-squares line
- Make a residual plot

Fathom Skills: Students will learn how to

- Use a residual plot to recognize a better function model

Mathematics Prerequisites: Students should be able to transform data and straighten data; use a linear equation in slope-intercept form (specifically, a least-squares line); interpret the coefficient of variation (r^2) for a least-squares line; use residuals to judge how well a function fits data; and recognize problem situations that result in quadratic functions.

Mathematics Skills: Students will learn how to use residuals to recognize a better function model and find a linear model that fits transformed data, and work backward to find a nonlinear model that fits the original data.

General Notes: This activity is about finding a model for nonlinear—specifically quadratic—data. However, unlike Moore's Law and Printing Paragraphs, both of which used sliders to improve a curved function with the original data, this activity looks for ways to straighten the data and then uses a least-squares line with the transformed data. This activity also extends students' understanding of residual plots. In the previous two activities, students used residual plots to improve function models that were apparently not perfect. Conversely, in this activity, the first model (a linear model) looks almost perfect when you graph it—it even has $r^2 = 1.00$—but when you look at the residuals, you know something is not quite right.

The task, basically, is to figure out how to use the diameter of a roll of adding-machine paper to tell how much paper is on the roll. Students do this by pulling off some paper and finding a relationship between the diameter and how much is pulled off. Once they find this relationship, they can calculate how much paper was on the roll by assuming the diameter is reduced to the core of the roll. The function is nonlinear because you use up more diameter per meter of paper as you get closer to the core—you go around more times. Furthermore, the function is quadratic because you're dealing with the area of an annulus. (See Extension 2 for a full explanation.)

Careful and precise measurements are imperative to the success of this activity. Students should be particularly careful measuring the diameter in millimeters, even going to the extent of approximating half- or quarter-millimeters. With careless measurements, the original data may not show the requisite bowing effect. If students do not clearly see the bowing in their own residual plot, point them to the sample graph on the student worksheet and/or have them work with the sample document **HowMuchPaper.ftm.** This document has sample diameter and length measurements from a 130-foot roll (approximately 39.6 meters) with a 22-millimeter core. (The sample answers that follow are based on the data in this document.) However, you are strongly encouraged to let students collect their own measurements and learn from their mistakes. If you can spare the time, you may prefer to use poorly collected data as a lesson in the necessity for precise measurement and then allow students to repeat the data collection with new rolls.

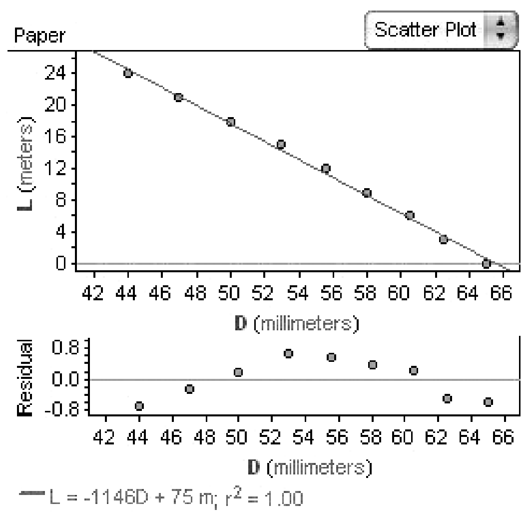
This activity is most appropriate for students in Algebra 2 (and preferably students who have had a Geometry course) because the function is quadratic and is based on the area formula of an annulus. However, you can successfully use it with Algebra 1 students because they don't have to understand the algebra and geometry of the situation to get good math out of the activity. Students in Algebra 2, Geometry, Precalculus, and Statistics can all be challenged by looking at the algebraic derivation of the relationship (Extension 2) and/or how to go from the linear function for the straightened data to a quadratic function for the original data (Explore More 2).

MAKE A CONJECTURE

Q1 Students usually agree that the function will curve, but they disagree about which way it will curve. All students should make a sketch that is decreasing—as diameter increases, length should decrease.

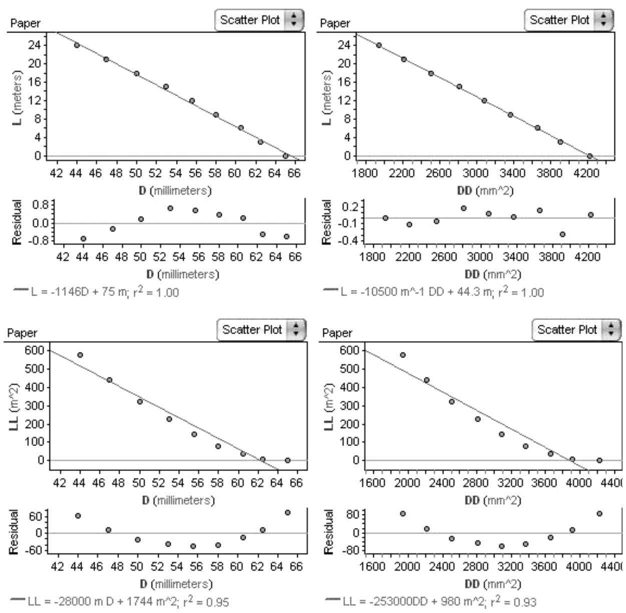
INVESTIGATE

Q2 Some students might feel that the least-squares line is a good fit because the residuals are very close to zero. Others might feel that it is not a good fit because of the bowing.



Q3 Students are likely to say that the decreasing slope of the least-squares line meets their expectations—that as diameter increases, length decreases. On the other hand, students are likely to say that the least-squares line contradicts their expectations because they predicted something curved.

Q4 The best graph is *L* versus *DD*, because the residuals have no pattern (or at least no prominent pattern) and they have the smallest magnitudes.

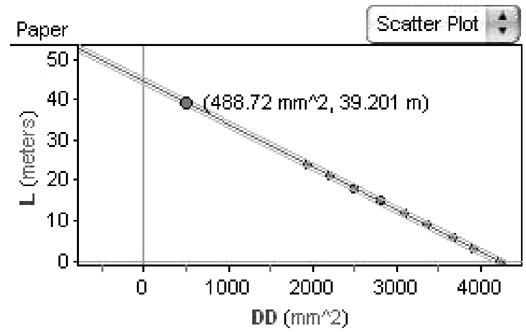


Q5 $L = -10500\ \text{m}^{-1}\ DD + 44.3\ \text{m}$

Q6 For these sample data, the core was $D = 22$ mm. To substitute this into the equation in Q5, which is stated with meters as the units, you need to square and convert to square meters: $DD = 0.000484\ \text{m}^2$.

Q7 For these sample data:
$L = -10500\ \text{m}^{-1}\ (0.000484\ \text{m}^2) + 44.3\ \text{m} =$
$-5.082\ \text{m} + 44.3\ \text{m} \approx 39.2\ \text{m}$.

Note: Rather than substituting into the function, students could trace the graph to find the value of *L* when *DD* is approximately 484 mm^2. (Students probably won't be able to trace to exactly 484 mm^2.) That way, students don't need to do a unit conversion from square millimeters to square meters, although they still need to square the diameter of the core. This also gives a value around 39.2 m.



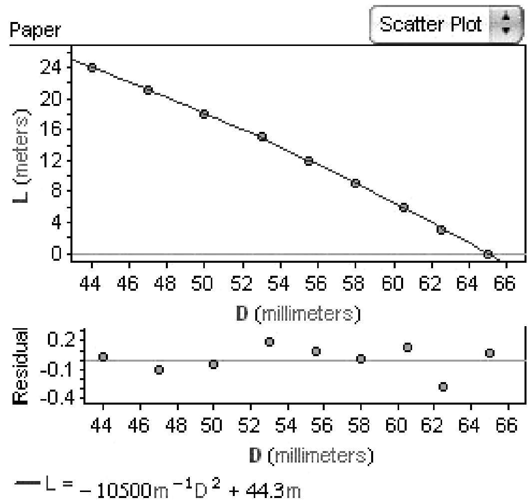
Q8 The roll used for the sample answers was advertised as 130 ft. Converting 39.2 m to feet gives approximately 129 ft. Conversely, if the roll had a full 130 ft, the answer to Q7 should have been about 39.6 m. So, the prediction in Q7 was pretty good.

Q9 The reason that the data looked fairly linear is a matter of perspective. If you zoom in close enough, any curve can begin to look straight. In terms of this geometric situation, a few meters of paper wraps around the roll only a few times when the roll is full, so the diameter changes only a little. However, if you had collected more data and gotten closer to the core, you would notice the curvature more prominently. That is, when you are near the core, the same few meters of paper will wrap around the roll several times and the diameter will change a lot.

EXPLORE MORE

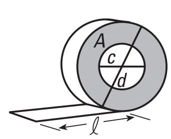
1. The roll used for the sample answers had a red band at diameter $D = 25$ mm, or $DD = 0.000625\ \text{m}^2$. This means that $L = -10500\ \text{m}^{-1}\ (0.000625\ \text{m}^2) + 44.3\ \text{m}$, or approximately 37.7 m have been removed. So there are 39.2 – 37.7, or about 1.5 m left.
2. Transforming the linear function into a quadratic function is fairly simple—just replace *DD* with D^2. Getting a quadratic function makes sense because removing paper changes the area of the circle, and the area of a circle is found by squaring the diameter. If students plot the quadratic function, it will fit the points better than the linear function in step 6 and

will give the same residual plot that the transformed data gave in step 9. To plot the function, students must include units on the coefficient and constant.



EXTENSIONS

1. Have students repeat the activity with a roll of toilet paper and come up with a way to determine the number of squares left on the roll as a function of its diameter. Toilet paper is "squishier" than adding-machine paper and therefore has more variability, so more experience and care are needed to get the activity to work. Students will need to devise creative ways to measure the diameter and length carefully and precisely.
2. Have students use algebra and geometry to derive the theoretical relationship between diameter and length. They can find a formula for either cumulative length pulled off as a function of diameter, or the length *remaining* on the roll as a function of diameter. The two formulas are closely related.

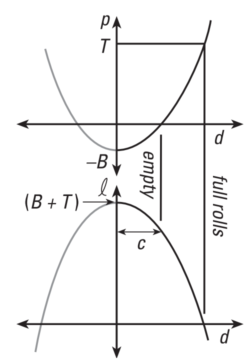
   The length of paper remaining on the roll (p) is proportional to the cross-sectional area of the roll. Because the roll has a core in the middle, that shape is an annulus. The area of an annulus is the area of the outside circle (with diameter d) minus the area of the core (with diameter c). If you let the constant of proportionality be K—which means the thickness of the paper is $\frac{1}{K}$—you start with the formula

$$p = KA_{\text{annulus}} = K\left[\pi\left(\frac{d}{2}\right)^2 - \pi\left(\frac{c}{2}\right)^2\right] = \frac{K\pi}{4}d^2 - \frac{K\pi}{4}c^2$$

Because the core does not change, the only independent variable in this function is d. Hence, the function is a quadratic function in the form $p = Ad^2 - B$. If you graph it as a parabola, it opens up with a vertex at $(0, -B)$, or $\left(0, \frac{K\pi}{4}c^2\right)$, and it crosses the horizontal axis where $\frac{K\pi}{4}d^2 - \frac{K\pi}{4}c^2 = 0$, or $d = c$.

In this activity, however, you don't know p, the paper remaining on the roll; you know ℓ, the amount you have pulled off. However, the sum of the amount remaining on the roll and the amount pulled off equals the constant total length that was originally on the roll, or $\ell + p = T$. Substituting $T - \ell$ for p and solving gives the formula

$$\ell = -Ad^2 + (B + T) \text{ or } \ell = -\frac{K\pi}{4}d^2 + \left(\frac{K\pi}{4}c^2 + T\right)$$



If you plot ℓ as a function of d^2 instead of as a function of d, you can see from the last equation that you'll get a straight line with $-A$, or $-\frac{K\pi}{4}$, as the slope and $(B + T)$, or $\left(\frac{K\pi}{4}c^2 + T\right)$, as the intercept. That's exactly what happened when you used a least-squares line on the transformed data of L versus DD.

It's worth noting that B, or $\frac{K\pi}{4}c^2$, is the amount of paper that could fill the core if the paper were wound down to zero radius.

COMPOUND INTEREST (PAGE 53)

Activity Time: 30–60 minutes

Required Document: CompoundInterest.ftm

Fathom Prerequisites: Students should be able to

- Adjust a slider (by dragging it and setting it to a specific value)
- Read cases from a case table

Fathom Skills: Students will learn how to

- *Optional:* Create a collection of measures (see Explore More 3)
- *Optional:* Add cases to a case table (see problem g on the next page)

Mathematics Prerequisites: Students should be able to use exponential functions for growth and decay problems. *Optional:* Students should be able to calculate simple and/or compound interest.

Mathematics Skills: Students will learn how to recognize how parameters affect a function.

General Notes: Compound interest is a pervasive feature of our society. Savings accounts, investments, retirement accounts, credit cards, mortgages, and automobile loans are all governed by the rules of compound interest. This activity uses a pre-made document that models compound interest. Students use sliders to set the parameters, and the case table and graph dynamically show how the account grows or decays. You can use this activity to introduce compound interest, supplement your textbook's discussion of this topic, and to teach valuable life skills.

Because everything in **CompoundInterest.ftm** is pre-made, the "feel" of this activity is somewhat different from the other activities in this book. The pre-made document also makes this an ideal activity to do as a teacher-led demonstration, especially if you do not have time to take the entire class to the computer lab. To save time, either as a student activity or as a class demonstration, don't forget that you can set the value of a slider by dragging it or by double-clicking the value of the slider and typing a specific value.

Within the document, a slider called *Principal* is used to represent only the amount of the original investment or loan. The case table then uses an attribute called *Balance* to represent the amount in the account at any subsequent time. In this way, only the first balance is based on the principal; each subsequent balance is compounded from the previous balance. Please note that accountants sometimes use the word *principal* to mean not only the original amount but also the current balance at the start of any compounding period.

For question Q5 on the worksheet, students might struggle with selecting the right case(s) to answer the question(s). The key to fully understanding the data in the case table is being able to understand the attributes *Year* and *Period*. *Year* represents the total number of years that have *elapsed*—notice that it starts with 0; likewise, *Period* is the total number of compounding periods that have elapsed. Hence, if students need to find the balance at the end of the 2nd compounding period of the 4th year, they would look for *Year* = 3 and *Period* = 2. That means 3 years have elapsed (you are now in the 4th year) and 2 whole compounding periods have elapsed. Similarly, if students need to find information for the end of 2 years, they should look for *Year* = 2, *Period* = 0, which means that 2 whole years have elapsed. Because of the potential confusion, it may be worthwhile to pause before Q5 and spot-check students' understanding of the attributes.

For brevity, question Q5 has only four options. The concept of each problem is slightly different and they are in order of increasing difficulty. If you would like to have students model more problems, here are three additional scenarios. You could also have students solve any compound interest problem from your current textbook.

e. You put $100 in a savings account with an annual rate of 6%, compounded monthly. You make no additional payments to the account. What is the balance of the account after 2 years? [Answer: $112.72]

f. You want to buy a used motorcycle for \$3500. Your aunt gives you \$500 to start, and you find a savings account that earns 9%, compounded monthly. How much will you need to add to the account each month in order to reach your goal by the end of 2 years? How much would you need to pay each month if your aunt had not given you \$500? [Answer: With aunt's gift: \$111 (or \$110.81 if *PaymentPerPeriod* is not restricted to integers); without aunt's gift: \$134 (or \$133.65)]

g. At age 25, Ben invests \$1000 in a retirement account. The account earns 5.5% interest, compounded quarterly. He plans to contribute another \$50 each quarter. Assuming the interest rate doesn't change, how much will he have if he retires at age 65? At 67? At 70? Do you think Ben will be able to retire "comfortably"? [Answer: At 65: \$37,584.42; at 67: \$42,343.02; at 70: \$50,530.49. None of these amounts would be enough to live "comfortably." *Note:* Students will need to add new cases to solve this problem.]

Because compound interest is an important application of exponents and exponential functions, some curricula introduce it as early as Prealgebra and the topic continues through Precalculus and Calculus. Hence, this activity can be used at a variety of grade levels, and the complexity of the activity can be varied for students with different levels of experience with the topic.

INVESTIGATE

Q1 Most importantly, *Principal* moves the vertical intercept of the graph to the value of *Principal,* regardless of the other sliders. The other points move too, but the amount by which they move depends on the settings of the sliders. Some students might describe the transformation due to *Principal* as a "translation" because the points appear to slide. However, because *Principal* is a constant factor, the transformation is actually a vertical dilation relative to the horizontal axis.

Q2 As *InterestRate* increases in the positive direction, the curve is stretched vertically and the steepness increases—the function grows more quickly. For negative values of *InterestRate,* the graph curves downward. A negative interest rate is equivalent to decay or depreciation. One real-world example is the depreciating value of a new automobile. Some students might describe the transformation due to *InterestRate* as a "dilation." However, because *InterestRate* is not a factor of the function, the transformation is not that simple. Furthermore, it is not a simple dilation because negative values of *InterestRate* do not create mirror images of similar positive values.

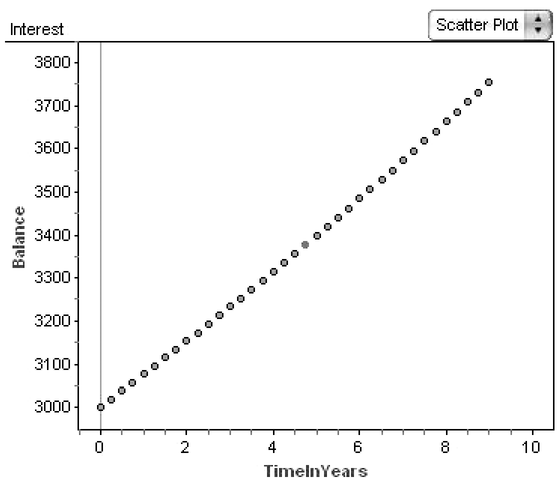
Q3 For positive interest rates, as *PeriodsPerYear* increases the points get closer together and the curve appears to get steeper—the function grows more quickly. For negative interest rates, as *PeriodsPerYear* increases the points get closer together and the curve appears to get flatter—the function decays more slowly.

Q4 As *PaymentPerPeriod* increases in the positive direction, the curve is stretched vertically and the steepness increases. For negative values of *PaymentPerPeriod,* depending on the value of *InterestRate,* the curve may be reflected and stretched across the line *Balance* = *Principal.* A negative payment means that you are taking money out of a savings account or that you are making a payment on a loan.

Q5 a. \$3377.02. Set *Principal* = 3000, *InterestRate* = 2.5, *PeriodsPerYear* = 4, and *PaymentPerPeriod* = 0. Find case 20, which represents the state of the investment after 4 full years and 3 quarters, or the 3rd quarter in the 5th year. *Note:* Many students may incorrectly find case 24 (*Year* = 5, *Period* = 3), but that actually represents the state of the investment after 5 full years, which is actually in the 6th year.

Interest

	Year	Period	TimeInYears	Balance	Interest	Payment
13	3	0	3	3232.90	20.21	0.00
14	3	1	3.25	3253.10	20.33	0.00
15	3	2	3.5	3273.44	20.46	0.00
16	3	3	3.75	3293.89	20.59	0.00
17	4	0	4	3314.48	20.72	0.00
18	4	1	4.25	3335.20	20.84	0.00
19	4	2	4.5	3356.04	20.98	0.00
20	4	3	4.75	3377.02	21.11	0.00
21	5	0	5	3398.12	21.24	0.00
22	5	1	5.25	3419.36	21.37	0.00
23	5	2	5.5	3440.73	21.50	0.00
24	5	3	5.75	3462.24	21.64	0.00
25	6	0	6	3483.88	21.77	0.00



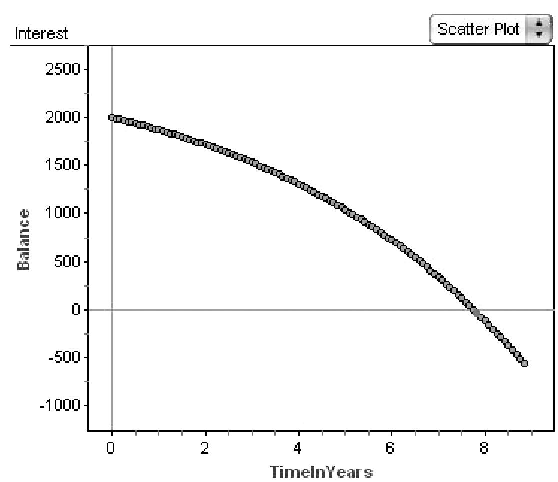
b. \$2683.08. Set *Principal* = 0, *InterestRate* = 7, *PeriodsPerYear* = 12, and *PaymentPerPeriod* = 100. Find case 26, which represents the state of the investment for the 25th paycheck. *Note:* Many students may incorrectly set *Principal* = 100, but you start with nothing until the first paycheck deducts something.

c. \$36 (or \$35.70 if *PaymentPerPeriod* is not restricted to integers). Set *Principal* = 1200, *InterestRate* = 4.5, and *PeriodsPerYear* = 12. Adjust the slider for *PaymentPerPeriod* into the negative range until case 37 is 0. *Note:* In order for the balance to decrease, *PaymentPerPeriod* must be negative. That's what the hint on the student worksheet is suggesting.

d. The balance will be fully paid off with the 94th payment, or after 7 years 10 months; Kelly will have paid 94 • 40, or \$3760. Set *Principal* = 2000, *InterestRate* = 18, *PeriodsPerYear* = 12, and *PaymentPerPeriod* = −40. Add at least 58 cases to the collection. Skim the case table or graph to find when *Principal* first passes 0. *Note:* Because the last payment actually makes the principal negative, in the real world Kelly's last payment would not be a full \$40; rather she would pay just the \$4.41 remaining on the previous bill.

Interest

	Year	Period	TimeInYears	Balance	Interest	Payment
84	6	11	6.91667	372.68	5.59	-40.00
85	7	0	7	338.27	5.07	-40.00
86	7	1	7.08333	303.35	4.55	-40.00
87	7	2	7.16667	267.90	4.02	-40.00
88	7	3	7.25	231.92	3.48	-40.00
89	7	4	7.33333	195.40	2.93	-40.00
90	7	5	7.41667	158.33	2.37	-40.00
91	7	6	7.5	120.70	1.81	-40.00
92	7	7	7.58333	82.51	1.24	-40.00
93	7	8	7.66667	43.75	0.66	-40.00
94	7	9	7.75	4.41	0.07	-40.00
95	7	10	7.83333	-35.53	-0.53	-40.00
96	7	11	7.91667	-76.06	-1.14	-40.00



EXPLORE MORE

1. In the long run, if your interest is compounded more frequently, your balance will be greater. Students may have noticed this from the steepness of the curve as they explored the effect of *PeriodsPerYear* for question Q3. Or they might actually set specific values for *Principal* and *InterestRate* and see what the balance is after, say, 1 full year of compounding under each scenario.

   Students may be surprised by the results of this extension. Before using Fathom to explore the situation, students might incorrectly assume that compounding once a year with an interest rate and compounding twelve times a year with one-twelfth the same interest rate should be equivalent, but they are not.

   You might want to introduce *annual percentage yield,* which is a banking term for the percent interest that actually accrues over one year when the interest rate is compounded more frequently than once a year.

2. At the least, students should recognize that the doubling time will decrease as the interest rate increases, because a higher interest rate means that you'll earn more money more quickly.

   If students collect coordinate pairs in the form (*interest rate, doubling time*), they may see that the general shape is similar to an inverse-variation function. You can also challenge students to do one or both of the mathematical derivations of the relationship, where P represents the principal in dollars, r represents the annual interest rate compounded c times per year, and t represents the number of periods required for doubling. If students have learned about inverses, you could discuss how and why the functions for interest rate and for doubling time are inverses of each other.

   Doubling time as a function of interest rate:

$$P\left(1+\frac{r}{c}\right)^t = 2P$$
$$\left(1+\frac{r}{c}\right)^t = 2$$
$$\log\left(1+\frac{r}{c}\right)^t = \log 2$$
$$t\log\left(1+\frac{r}{c}\right) = \log 2$$
$$t = \frac{\log 2}{\log\left(1+\frac{r}{c}\right)}$$

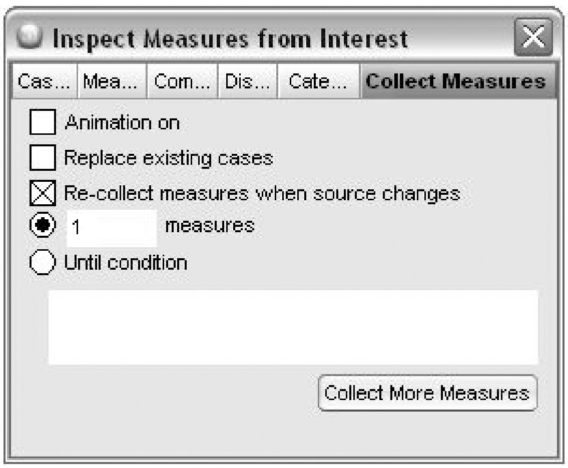
   Interest rate as a function of doubling time:

$$P\left(1+\frac{r}{c}\right)^t = 2P$$
$$\left(1+\frac{r}{c}\right)^t = 2$$
$$1+\frac{r}{c} = 2^{1/t}$$
$$\frac{r}{c} = 2^{1/t} - 1$$
$$r = c(2^{1/t} - 1)$$

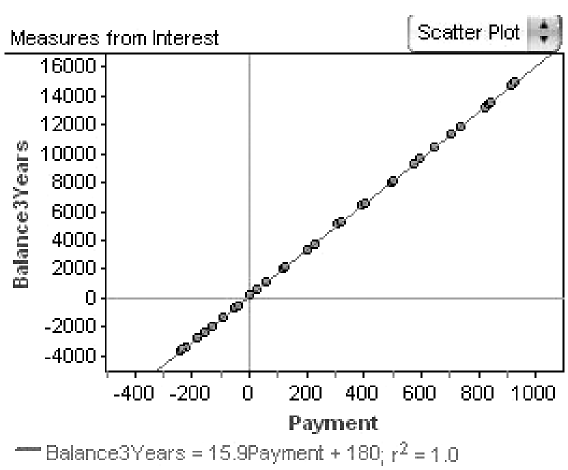
3. At the least, students should be able to manually record coordinate pairs for the value of *PaymentPerPeriod* beside the resulting value of *Balance* after 3 years (when *Year* $= 3$ and *Period* $= 0$). By looking at the table of coordinate pairs, or by creating a case table in Fathom and making a scatter plot, they should see that the relationship is perfectly linear.

   If you choose, this exploration can be used to introduce Fathom's ability to create a collection of measures. The pairs of values that you need to explore are any value of *PaymentPerPeriod* and *Balance* when *Year* $= 3$ and *Period* $= 0$. Show the collection inspector for *Interest* (double-click the collection) and go to the **Measures** panel. Create two attributes, *Payment* and *Balance3Years,* defined by the formulas *PaymentPerPeriod* and *lookupValueByIndex* ("Interest," "Balance," *PeriodsPerYear* • 3 + 1), respectively. These measures tell you the payment and balance after 3 years for any particular setting of *PaymentPerPeriod.*

Close the inspector, select the collection, and choose **Collect Measures** from the **Collection** menu. A derived collection called Measures from Interest is created; by creating a case table, you'll see that it contains 5 identical cases for *Payment* and *Balance3Years.* Show the collection inspector for this new collection and go to the **Collect Measures** panel. Choose the settings shown below.



Now drag the slider for *PaymentPerPeriod* and watch Fathom populate the case table. Make a scatter plot of *Balance3Years* versus *Payment* to see that there is a perfect linear relationship. (The graph below is for *Principal* = 100, *InterestRate* = 20, and *PeriodsPerYear* = 4.)



In an attempt to be completely quantitative, have students write down the equation of the least-squares line and values of all the parameters and then repeat for new settings of *Principal, InterestRate,* and/or *PeriodsPerYear.* (The easiest way to start over is to adjust the sliders, select all of the cases in the derived collection and delete them, and then start dragging *PaymentPerPeriod.*) Challenge students to find where the linear equation's slope and intercept come from. The vertical intercept of the linear relationship is the easiest to determine. Because the intercept occurs when *Payment* = 0, the balance after 3 years is simply $P\left(1+\frac{r}{c}\right)^{3c}$, where P is the principal, r is the interest rate, and c is the number of compoundings per year.

The slope, however, requires knowledge about recursion and geometric series. Because the payment occurs after the principal has accrued interest, the balance after t periods looks like

$$\left[\left[\left[\left[P\left(1+\frac{r}{c}\right)+\textit{Payment}\right]\left(1+\frac{r}{c}\right)+\textit{Payment}\right]\left(1+\frac{r}{c}\right)+\textit{Payment}\right]\cdots\left(1+\frac{r}{c}\right)+\textit{Payment}\right]$$

or

$$P\left(1+\frac{r}{c}\right)^{3c}+\textit{Payment}\left(1+\frac{r}{c}\right)^{3c-1}+\textit{Payment}\left(1+\frac{r}{c}\right)^{3c-2}+\cdots+\textit{Payment}$$

or

$$P\left(1+\frac{r}{c}\right)^{3c}+\textit{Payment}\left[\left(1+\frac{r}{c}\right)^{3c-1}+\left(1+\frac{r}{c}\right)^{3c-2}+\cdots+1\right]$$

The last expression is a linear expression in the form $A+Bx$. You'll notice that the constant term, $P\left(1+\frac{r}{c}\right)^{3c}$, is the same vertical intercept found previously. Because payment is the independent variable in the function, $\left[\left(1+\frac{r}{c}\right)^{3c-1}+\left(1+\frac{r}{c}\right)^{3c-2}+\cdots+1\right]$ is the slope. This, however, is a geometric series with first term 1, ratio $\left(1+\frac{r}{c}\right)$, and $3c$ terms, so it could be rewritten

$$\frac{1-\left(1+\frac{r}{c}\right)^{3c}}{1-\left(1+\frac{r}{c}\right)}$$

So, for any principal P and interest rate r compounded c times per year, the relationship between payment and balance after 3 years is

$$Balance3Years = \frac{1-\left(1+\frac{r}{c}\right)^{3c}}{1-\left(1+\frac{r}{c}\right)} \cdot Payment + P\left(1+\frac{r}{c}\right)^{3c}$$

You can now confirm the linear function in the preceding graph, using $Principal = 100$, $InterestRate = 20$, and $PeriodsPerYear = 4$:

$$Balance3Years = \frac{1-\left(1+\frac{0.20}{4}\right)^{3.4}}{1-\left(1+\frac{0.20}{4}\right)} \cdot Payment +$$

$$100\left(1+\frac{0.20}{4}\right)^{3.4} \approx 15.9 \cdot Payment + 180$$

Some students may recognize that the function can be further generalized for t years:

$$Balance3Years = \frac{1-\left(1+\frac{r}{c}\right)^{tc}}{1-\left(1+\frac{r}{c}\right)} \cdot Payment + P\left(1+\frac{r}{c}\right)^{tc}$$

JUPITER'S MOONS (PAGE 59)

Activity Time: 30–45 minutes

Required Document: JupiterMoons.ftm

Fathom Prerequisites: Students should be able to

- Open a document
- Create attributes defined by formula
- Create graphs of two attributes
- Create a slider

Fathom Skills: Students will learn how to

- Rescale a graph by using the pop-up menu to choose the graph type again
- Highlight points in graphs by selecting them in the case table
- Find the coordinates of a particular point in a graph
- Plot a function in a graph

Mathematics Prerequisites: Students should be able to write the equation for a translated and dilated periodic function.

Mathematics Skills: Students will learn how to transform data to improve a model; create a formula for the phase of a periodic function; write a periodic function to model data; and vary the period, phase, and amplitude of a periodic function to model data.

General Notes: This activity is about accuracy. How accurately can you determine how long it takes one of the moons of Jupiter to go around that planet? Students will propose one period and overlay the waves to see whether they line up. This requires a good understanding of the *phase* of a periodic function. Below left is a diagram of the path of a moon, with a proposed period—but the period is too short. If we overlay the periods, it looks like the diagram below right.