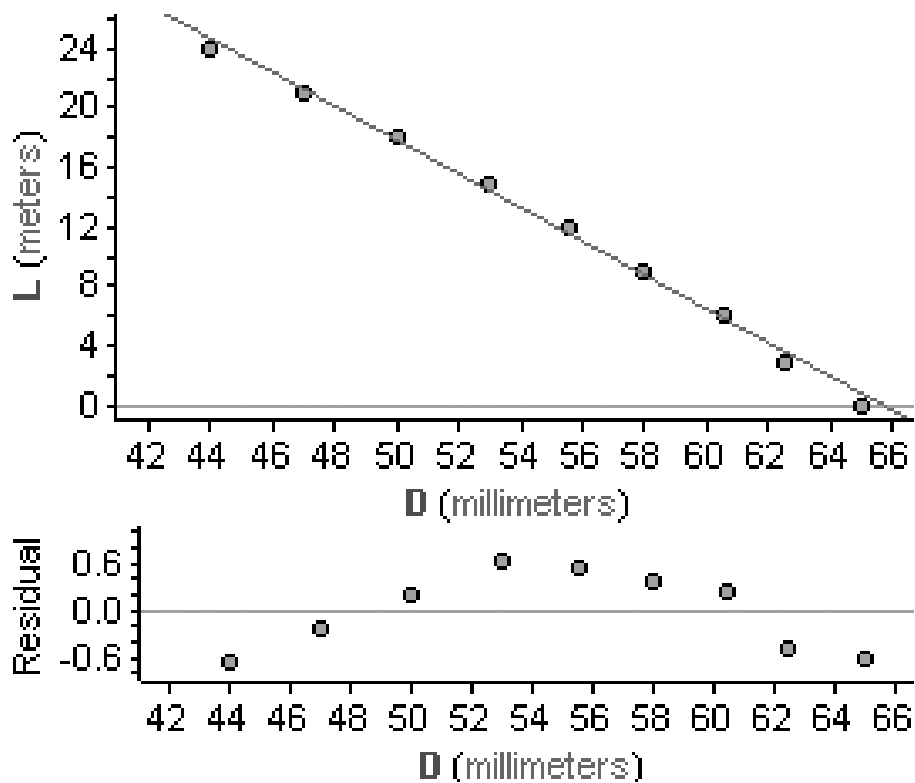


# Algebra 2



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Mauna Loa is a volcano on the island of Hawai'i. Since 1958, air samples have been continuously collected at the Mauna Loa Observatory and analyzed for carbon dioxide ( $\text{CO}_2$ ) concentrations.

You may know that  $\text{CO}_2$  is one of the “greenhouse gases” that scientists monitor in relation to global warming. The Mauna Loa Observatory is very important to the study of global warming. It is far removed from industry and other human activity, so any changes in  $\text{CO}_2$  there are good indicators of global changes in the Earth's atmosphere.

In this activity, you'll look at the Mauna Loa data and see if there are any patterns.

## MAKE A CONJECTURE

You probably already have some ideas about what the data look like, especially if you know anything about global warming. Before you begin, think about what you expect to see.

- Q1** Since 1958, do you think the concentration of  $\text{CO}_2$  at Mauna Loa has been increasing, decreasing, or staying the same?
- Q2** If you think the concentration has been increasing or decreasing, what shape do you think the data will have? Will it be straight or curved?

## INVESTIGATE

Now you'll use Fathom to look at the Mauna Loa data. Keep questions Q1 and Q2 in mind, because along the way you'll see whether your predictions were correct.

- 1.** Open the document **MaunaLoa2003.ftm**. You'll see two *collections* of data that look like boxes of gold balls: Mauna Loa, Continuous and Mauna Loa, By Month. Both collections contain the average  $\text{CO}_2$  concentrations in parts per million (ppm) for every month from 1958 to 2003, but they are organized differently. You'll also see two *case tables*, one for each collection, that help you view the data in the collections. Each row is a *case* and each column is an *attribute*.
- 2.** Scroll through the case tables and make sure you understand how the data are organized. (*Note:* There are seven months that have no data: four in 1958 and three in 1964. You'll see these as empty cells.)
- Q3** What was the  $\text{CO}_2$  concentration in March 1970? How do you find the answer from Mauna Loa, Continuous? How do you find the answer from Mauna Loa, By Month?

- Q4** From looking at the case tables, is the concentration of  $\text{CO}_2$  increasing, decreasing, or staying the same? Explain your answer, as well as any difficulty you may have had.

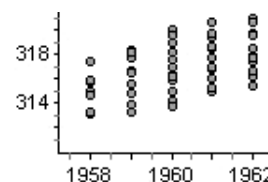
Looking for patterns in a case table isn't very efficient when you have a lot of data. Graphs are sometimes a more efficient way to look at the data.



- 3.** Drag a new graph from the shelf. An empty graph appears.
- 4.** Go to the case table for Mauna Loa, Continuous and drag the attribute *Year* from the case table to the graph. Drop the attribute below the horizontal axis. Then drag *CarbonDioxide* to the vertical axis. The graph becomes a scatter plot that shows the concentration of  $\text{CO}_2$  over time. If your graph of points is bunched up, drag the corner of the graph to make it larger.

- Q5** From looking at the graph of *CarbonDioxide* versus *Year*, is the concentration of  $\text{CO}_2$  increasing, decreasing, or staying the same?

You probably recognize a problem in your graph. Although the graph shows changes over time, you have big gaps between each year. The problem is that *Year* doesn't account for *Month*. You'll fix that in the next few steps.



- 5.** In the case table for Mauna Loa, Continuous, scroll to the far right. Click *<new>* in the last column, type *YearAndMonth*, and press Enter. This creates a new attribute.
  - 6.** Select the *YearAndMonth* column and choose **Edit Formula** from the **Edit** menu. The formula editor appears. Type a formula that appropriately combines *Year* and *Month* and click **OK**. The column fills with calculated values.
- Q6** What formula did you use? Use at least one case in the case table to verify that the formula is working properly.
- 7.** Drag *YearAndMonth* from the case table and drop it below the horizontal axis of the scatter plot. The new attribute should replace the old one.
- Q7** From looking at the graph of *CarbonDioxide* versus *YearAndMonth*, is the concentration of  $\text{CO}_2$  increasing, decreasing, or staying the same? Describe any other patterns that you see in the data. (You may need to make the graph larger to see a pattern.)
- Q8** How would you describe the shape of the data? How does the shape compare to what you predicted in Q2?

Because linear equations are simple—they have only a slope and an intercept—lines are frequently used to “fit” data. Fathom gives you three ways to fit a line to your data: a movable line, a median-median line, and a least-squares line.

- 8.** Select your graph and choose one of the lines of fit from the **Graph** menu. Try a movable line first because it allows you to adjust the line to the fit that you think is best. If you’ve learned about median-median lines or least-squares lines in class, your teacher may tell you to try those lines of fit, too.
- Q9** How well does your line fit the data? What other mathematical curves do you think might fit the data better?
- Q10** Based on your line of fit, approximate the rate at which the concentration of  $\text{CO}_2$  is increasing or decreasing.

For these last two questions, you can use the graph that you’ve already made or make a new graph. You can use the data in either collection, and you can make new attributes if you want. Because different people will use different strategies, fully explain how you found each answer.

- Q11** Predict the general  $\text{CO}_2$  concentration at Mauna Loa this year. (*Hint:* If you used a median-median line or a least-squares line, you can trace it.)
- Q12** Predict what the  $\text{CO}_2$  concentration will be on your 21st birthday.

## EXPLORE MORE

- 1.** When you used the attribute *YearAndMonth*, you probably noticed that the data have seasonal fluctuations. And when you used a line of fit, you probably noticed that the general shape of the data is slightly curved. Use what you know about other functions, including periodic functions, to find curves that fit the data better. Once you have a better fit, answer Q11 and Q12 with your new function. (*Hint:* Use Fathom Help to learn about plotting functions. You may also want to learn about sliders.)
- 2.** Focus on the seasonal fluctuations for each year. In general, during which month does the maximum  $\text{CO}_2$  concentration occur? The minimum? Is the range of the fluctuations increasing, decreasing, or staying the same over time? Describe any other patterns that you see in the seasonal fluctuations.

# Function Transformations

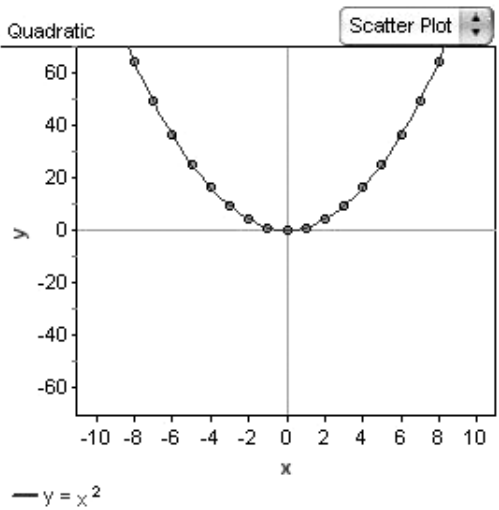
In this activity, you'll study data that perfectly fit a function. The functions you'll look at may already be familiar—quadratic, absolute value, exponential, square root, inverse variation, and linear. By changing the data, you'll explore how the equation of the function changes.

## INVESTIGATE

1. Open the document **FunctionTransformations.ftm**. You'll see a collection and case table of  $x$ - and  $y$ -coordinates. You'll also see a scatter plot of the coordinate pairs  $(x, y)$ . Notice that the points form a parabola.

You probably know that a parabola is formed by a quadratic equation. In fact, the  $y$ -coordinates are calculated by the formula  $y = x^2$ .

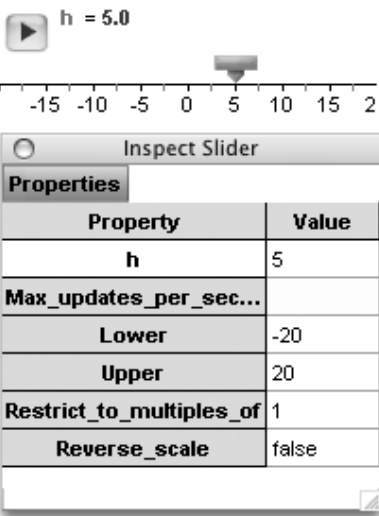
2. Select the graph and choose **Plot Function** from the **Graph** menu. In the formula editor, type  $x^2$  and click **OK**. You should get a parabola that perfectly fits the points.



You are now going to explore what happens when you add the same amount to each  $x$ - or  $y$ -coordinate. You'll use a slider that allows you to easily explore the effect of adding different amounts.

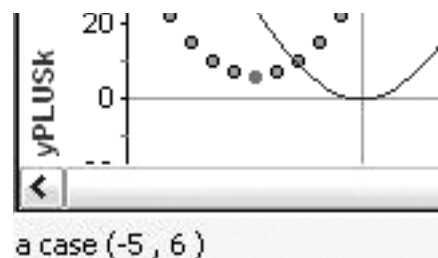


3. Drag a new slider from the shelf.  
The slider will be called  $V1$ . Drag the slider thumb and notice how the value of  $V1$  changes to different decimal values. Click the green arrow button to animate the slider; click the button again to stop it.
4. Double-click near the slider's axis to show the slider's inspector. Enter the values shown here, which rename the slider  $h$  and limit its values to integers between  $-20$  and  $20$ . Close the inspector and drag the slider thumb to test its behavior.



5. Select slider  $h$  and choose **Duplicate Slider** from the **Object** menu. Rename this slider  $k$ , but keep it restricted to integers between  $-20$  and  $20$ .
6. In the case table, create a new attribute,  $xPLUSh$ . Define it with a formula that adds the value of  $h$  to each  $x$ -coordinate.
7. Make another new attribute,  $yPLUSk$ , that adds  $k$  to each  $y$ -coordinate.
8. Select the scatter plot of  $(x, y)$  and choose **Duplicate Graph** from the **Object** menu.
9. Select the duplicate graph and choose **Show Axis Links** from the **Graph** menu. A broken chain link will appear by each axis. Drag the broken chain link from the  $x$ -axis and drop it on the  $x$ -axis of the original graph. Do the same thing to link the  $y$ -axes. Now, if you change the axes in one graph, the axes change in the other graph, too.
10. So far, you have two identical scatter plots of  $(x, y)$ . Drag the attribute  $xPLUSh$  from the case table and drop it on the  $x$ -axis of the duplicate graph. Drag  $yPLUSk$  to the  $y$ -axis. You now have a scatter plot of  $(xPLUSh, yPLUSk)$ , but you still have the graph of  $y = x^2$  for reference.
11. Because the names of your axes have changed, the function is now called  $yPLUSk = x^2$ . Double-click the function equation at the bottom of the graph and edit it to show  $xPLUSh$  rather than  $x$ .
- Q1 Set the value of  $k$  to 0. Then drag slider  $h$  and watch the scatter plot change. How does the scatter plot of  $(xPLUSh, yPLUSk)$  compare to the scatter plot of  $(x, y)$ ? How do the points change when you add  $h$  to each  $x$ -coordinate?
- Q2 Set the value of  $h$  to 0. Then drag slider  $k$  and watch the scatter plot change. How does the scatter plot of  $(xPLUSh, yPLUSk)$  compare to the scatter plot of  $(x, y)$ ? How do the points change when you add  $k$  to each  $y$ -coordinate?
- Q3 Try changing both sliders and watch the scatter plot change. Use what you learn to predict what the scatter plot would look like for these values. After you write your prediction, use the sliders to check your work.
  - a.  $h = 3, k = 5$
  - b.  $h = -8, k = 0$
  - c.  $h = 5, k = -20$

- Q4** An important aspect of any parabola is its *vertex*. Set the sliders to any values and then point the cursor to the vertex. Look in the bottom-left corner of the Fathom window for the vertex's coordinates. How do the coordinates relate to the values of  $h$  and  $k$ ?



- Q5** The original parabola is  $y = x^2$ . What parabola fits the points transformed by  $h$  and  $k$ ? Make a guess and plot a new function. If it doesn't go through the points, double-click the equation and make changes in the formula editor. In general, what quadratic function fits the points  $(xPLUSh, yPLUSk)$ ? (*Hint: Try using  $h$  and  $k$  in the function equation and see whether the parabola moves with the points when you change the sliders.*)
- Q6** The transformations that you've just seen are called *translations*. You learned how to translate points and how to translate a quadratic function. Do the same rules apply to all functions? Below the case table are five other collections of points for  $y = |x|$ ,  $y = 2^x$ ,  $y = \sqrt{x}$ ,  $y = \frac{1}{x}$ , and  $y = x$ . Create case tables, attributes, graphs, and functions to explore translations with these functions. Summarize your findings by answering questions Q1–Q5 for each function. Then explain how translations were similar or different for all of the functions. In general, if you are given any function  $y = f(x)$ , how would  $h$  and  $k$  affect it?

You've seen what happens when you add  $h$  and  $k$ . Now you'll see what happens when you multiply  $x$  or  $y$  by a constant.

- 12.** Create two sliders,  $a$  and  $b$ , that each go from  $-10$  to  $10$  by multiples of  $0.25$ .
  - 13.** Create two new attributes,  $aTIMESx$  and  $bTIMESy$ , that multiply the values of the sliders by the coordinates.
  - 14.** Make a scatter plot of  $(aTIMESx, bTIMESy)$ . Be sure that you can compare the transformed points to the graph of  $y = x^2$ .
  - 15.** Adjust the sliders for  $a$  and  $b$  and watch how the scatter plot changes.
- Q7** How does the scatter plot change when you change  $a$  and  $b$ ? Fully describe the effects, including anything special that happens.
- Q8** What parabola fits the points transformed by  $a$  and  $b$ ? In general, what quadratic function fits the points  $(aTIMESx, bTIMESy)$ ?

- Q9** The transformations that you've just seen are called *dilations*. Do the same dilation rules apply to all functions? Explore dilations with the other collections and summarize your findings. In general, if you are given any function  $y = f(x)$ , how would  $a$  and  $b$  affect it?
- Q10** A special type of transformation occurs when  $a = -1$  or  $b = -1$ . Explore these values for any (or all) of the functions and summarize your findings. What kind of transformation happens when  $a = -1$ ? When  $b = -1$ ?

## EXPLORE MORE

1. Explore what happens when you combine translations and dilations. That is, look at scatter plots of ( $a$ *TIMES* $x$ *PLUS* $h$ ,  $b$ *TIMES* $y$ *PLUS* $k$ ). Summarize your findings. If you are given any function  $y = f(x)$ , how would  $a$ ,  $b$ ,  $h$ , and  $k$  affect it? If you see the graph of any parabola (or any absolute value function, or any exponential function, and so on), how could you use the points on the graph to determine the function's equation?
2. Another fundamental type of transformation is a *rotation*. Find ways to transform the coordinates to create rotations. How are rotations significantly different from the other function transformations that you've seen in this activity?



## Moore's Law

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Gordon Moore (b. 1929) is cofounder and past chief executive officer (CEO) of Intel Corporation, a company known for making computer central processing units (CPUs). In 1965, three years before founding Intel, Moore made a famous observation that computer processors would get more powerful exponentially. The press called his observation “Moore’s Law,” and different variations of it have been stated ever since.

One way to measure a CPU’s power is to count the number of transistors on it. In this activity, you’ll look at data about Intel’s processors. You’ll test Moore’s Law by using an exponential function to model the data, determine how frequently the number of transistors doubles, and predict the future of CPUs.

### INVESTIGATE

1. Open the document **Moore'sLaw2004.ftm**. You’ll see a collection and case table. There is one case for each major Intel CPU from 1970 to 2004. The attributes tell you the name of the CPU, the year it was introduced, the number of transistors, and the number of years since 1970. (*Note:* The year 1970 was chosen as the nearest multiple of 10 before the year of the first Intel processor.)
2. Make a scatter plot of *Transistors* versus *Since1970*.
- Q1 Sketch the graph and describe its shape in one or two sentences. What mathematical curve does the graph resemble? What does that mean about the number of transistors over time?
3. Moore’s original observation was that “complexity . . . has increased at a rate of roughly a factor of two per year.” If you assume that “complexity” can be measured by number of transistors, you get the function  $Transistors = 2^{Since1970}$ . Plot this function on your graph.
- Q2 Describe how well the function fits the scatter plot. How could you transform the function to make it fit the data better?
4. Make sliders and use them as parameters to transform the function  $Transistors = 2^{Since1970}$ . Try any transformations that you think are necessary—translations, dilations, or reflections. Adjust your sliders to get the best fit possible. (*Hint:* You may want to zoom in to check your fit. Select the graph. Hold down Ctrl (Win) Option (Mac) and click in the graph. Hold down Shift as well to zoom out.)

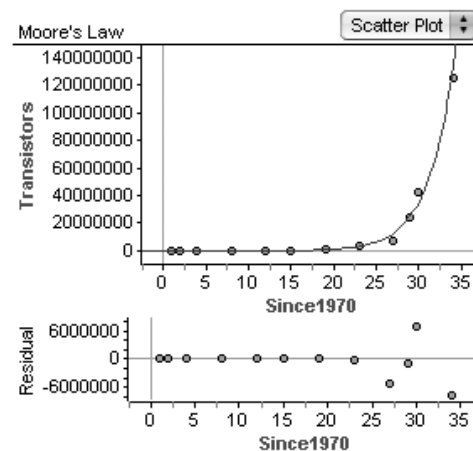
- Q3** List the sliders and transformations that you tried. Which ones helped you get the best fit? Are they the same transformations that you predicted in question Q2?
- Q4** Add the curve to your sketch in Q1. Record the final function, inserting the values of the sliders where necessary.
- Q5** According to your function, how many years does it take for the number of transistors to double? That is, what is the *doubling time*?
- 5.** Move the cursor over the graph of your function. The arrow cursor changes to a red point. Click and drag to trace the graph with the red point. Notice that the coordinates appear beside the point and in the bottom-left corner of the Fathom window. This gives you a way to approximate how many transistors a CPU would have in future years, according to your function model.
- Q6** Approximately how many transistors would a CPU have in 2005? In 2010?

A *residual* is the vertical distance between your function and an actual data point. A *residual plot* shows all of the residuals and helps you judge how well the function fits.

- 6.** Select the graph and choose **Make Residual Plot** from the **Graph** menu. A residual plot appears below your original graph.

- 7.** Adjust your sliders and watch how the residual plot changes as the function's fit changes.

- Q7** What happens to a residual when the function passes through the original data point? What happens when the function doesn't pass through the data point? What happens to the residual when the fit gets better? Gets worse?
- Q8** Which data points in your original graph have the most problematic residuals? Why did you pick these points?
- Q9** Use your sliders to find the best residual plot. Record this new function, inserting the values of the sliders where necessary. How does it compare to your function in Q4?



- Q10** According to your function, was Gordon Moore correct to observe that computer processors get more “complex” exponentially? Was he correct that the complexity doubles each year? How would you state Moore’s Law based on your function?
- Q11** What do you think about using your Moore’s Law function to make predictions for the future? Do you think transistors will continue to increase at the same exponential rate? Why or why not?

## EXPLORE MORE

1. Research any new Intel processors that have been released since the Pentium 4. Compare the processor’s actual number of transistors to the number of transistors that your function predicts for that year of release. How well did your model work?
2. You may have learned about logarithms. Create a new attribute that takes the logarithm of *Transistors*, and make a scatter plot of  $\log Transistors$  versus *Since1970*. Describe what happens and find a function to fit this new graph. How can you use this new graph to predict values for the future? How can you use the equation of the function of fit to get something like the exponential function you used before?

# Printing Paragraphs

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You will need

- The Seven Paragraphs worksheet
- centimeter ruler

You have a handout with seven paragraphs on it. They're all exactly the same text, but they are set in columns of different widths. As you might expect, when the paragraph is really wide, it is also really short. Conversely, narrow paragraphs are really long.

Exactly how does the height of the paragraph depend on the width? In this activity, you'll find a function that models the data. Then you'll use a residual plot to improve your model.

## MAKE A CONJECTURE

Before you begin, look at the seven paragraphs. Think about how the horizontal width and vertical length are related, and what you might see.

- Q1** If you make a graph of length versus width, what would it look like? Fully describe your prediction or make a sketch.
- Q2** Approximately what geometric shape is each paragraph?
- Q3** Each paragraph takes up area on the page. What geometric formula could you use to find the area of each paragraph?
- Q4** Do you think the area taken up by each paragraph is approximately the same? Explain your reasoning.

## EXPERIMENT

Now measure the horizontal width and vertical length of each paragraph to the nearest centimeter. First, decide *how* you will measure. The width is going to be particularly tricky because some lines are long and some are short. Be as precise and consistent as you can.

- 1.** Measure the width and length of each paragraph. Record your results in a table.
- Q5** Describe how you measured width and length. Where did you begin and end each measurement? Why did you choose the beginning and end that you did?

## INVESTIGATE

Next you'll analyze the data with Fathom.

- 2.** Start Fathom and create a new case table with the attributes *Width* and *Length*. Enter your data into the case table. Don't use units with your measurements.

3. Make a scatter plot of *Length* versus *Width*.

Q6 Sketch the graph and describe its shape in one or two sentences. How does it compare to your prediction in question Q1?

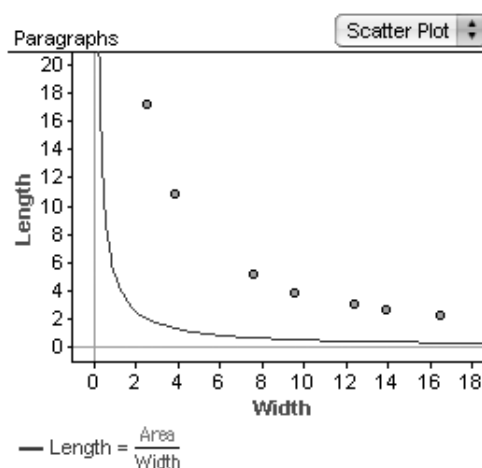
For Q2, you and your classmates probably said that the paragraphs are rectangles. And, for Q3, you probably thought about  $Length \cdot Width = Area$ . You can rewrite this formula as the inverse variation  $Length = \frac{Area}{Width}$ . If *Area* happens to be the same for all of the rectangles, then this gives you a functional relationship between *Length* and *Width*. You'll use a slider to explore *Area*.

4. Make a new slider. Name it *Area*.

5. Select the graph and plot the function  $Length = \frac{Area}{Width}$ . A curve appears but it probably doesn't fit the points at all.

6. Adjust the slider and try to make the curve fit the points. You'll probably need to adjust the slider's scale.

Q7 Describe any patterns that you notice while adjusting the slider. Explain why the patterns exist based on the printed paragraphs.



Q8 Do all of the paragraphs have exactly the same area? Do they have approximately the same area? Explain.

Q9 If you had to pick one value of *Area* that makes the curve fit the points best, what value would it be? Explain why you would select this value.

Q10 Adjust the slider to find the maximum and minimum values for *Area* that you think are reasonable. (*Hint:* To get finer control of the slider, you can zoom in to its axis by pressing Ctrl (Win) Option (Mac) while you click on the axis.)

7. In your case table, create a new attribute, *A* (for area), and define it with the formula  $Length \cdot Width$ .

Q11 Do the values of *A* support the patterns that you saw in Q7? Do they support the range of values that you saw in Q10? Do you see any additional patterns in the areas?

**Q12** How does your best value of *Area* from Q8 compare to the values of *A*?

A *residual* is the vertical distance between your function and an actual data point. A residual of zero means the function perfectly fits the point. A *residual plot* shows all of the residuals and can help you judge how well a function fits the data.

**8.** With the graph selected, choose **Make Residual Plot** from the **Graph** menu.

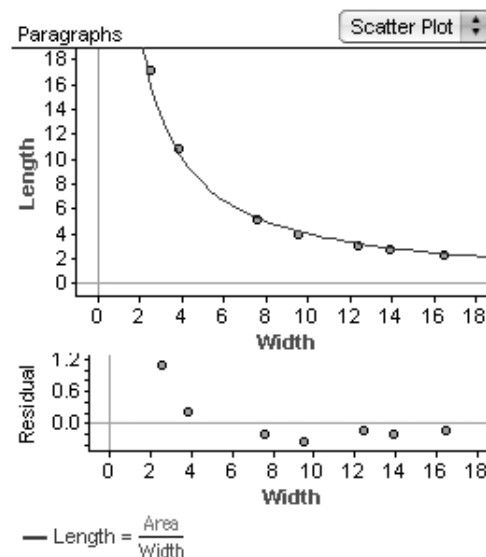
A residual plot appears below your graph.

**9.** Adjust the slider and watch how the residual plot changes as the function's fit changes. Based on the residual plot, find one value of *Area* that makes the curve fit the best.

**Q13** What value of *Area* did you get? Explain how you decided what value was best.

**Q14** How does this value of *Area* compare to the value you chose in Q9?

**Q15** From what you have done so far, how does the height of the paragraphs depend on the width? Do you think you've found the best model for this relationship? In what ways could it be improved, if any?



## EXPLORE MORE

**1.** You probably noticed that it is impossible to fit the curve to all seven data points. The points that represent the narrowest two paragraphs are particularly problematic. That's because the paragraphs aren't perfect rectangles—a line may or may not have extra white space at the end because of words moving to the next line. Assume that the average amount of white space is the same for every line. Make a new slider called *White*. Then the area of the paragraph is  $Area = Length \cdot (Width - White)$ . Modify your inverse variation function accordingly and adjust the sliders to find the best fit now. Does *White* help reduce the residuals? Describe which residuals *White* impacts the most, and explain why this makes sense. Is your value for *Area* bigger or smaller than before you added *White*? Explain why the change in *Area* makes sense.

2. After finding the best fit with *Area* and *White*, adjust the sliders some more and watch how the graph of the inverse-variation curve and the residual plot change. What happens to the curve when you change *Area*? What happens to the residuals? What happens to the curve when you change *White*? What happens to the residuals? Try to explain mathematically why this happens.
3. When you first made a graph of *Length* versus *Width*, you may have thought that the relationship was exponential. That would have been a good guess, especially if you didn't know the geometric relationship. (Mathematicians don't always know what model is best. They usually have to try several to find the best fit.) Try fitting an exponential function to the data. Use sliders to help you adjust parameters in the function, and use residual plots to help you judge how well the function fits. After you've found the best fit that you can with an exponential function, compare its residual plot to the residual plot for the best fit with the inverse-variation function. Which residual plot implies the better fit?

## THE SEVEN PARAGRAPHS

Some rational functions can create very different kinds of graphs from those you have studied previously. The graphs of these functions are often in two or more parts. This is because the denominator, a polynomial function, may be equal to zero at some point, so the function will be undefined at that point. Sometimes it's difficult to see the different parts of the graph because they may be separated by only one missing point, called a hole. At other times, you will see two parts that look very similar—one part may look like a reflection or rotation of the other part. Or you might get multiple parts that look totally different from each other.

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from *Discovering Advanced Algebra* by Murdock, Kamischke, and Kamischke



# How Much Paper Is Left?

You will need

- full roll of adding-machine paper
- millimeter ruler
- meterstick

You have a full roll of adding-machine paper. The question for this activity is “How much paper is on the roll?” The catch is that you can’t find the length by unrolling the entire roll.

As you pull paper from the roll, the roll gets smaller. Because the roll is a circle (or a cylinder, to be exact), you can use the diameter as one way to measure the roll’s size. This leads to a related question: “How does the diameter of the roll relate to how much paper you pull off?” Once you find a mathematical model for this relationship, you will be able to calculate the total length without unrolling it.

## MAKE A CONJECTURE

Before you start unrolling, make a conjecture about what will happen. You already know that as the diameter gets smaller, it means you’ve pulled off more paper. But what is the functional relationship?

**Q1** If you make a graph of the *cumulative* length pulled off ( $L$ ) as a function of diameter ( $D$ ), what would it look like? Is the function linear? Is it curved? If it’s curved, how does it curve? Sketch your prediction.

## EXPERIMENT

Now you’ll collect data by measuring diameter and cumulative length pulled off. For each measurement, be as careful and precise as you can.

1. Use your ruler to measure the diameter of the full roll to the nearest half millimeter. You haven’t pulled off any paper yet, so record this diameter for  $D$  along with 0 m for  $L$  in this table:
2. Pull off a few meters of paper. Measure the length with your ruler or meterstick and record it for  $L$ . Precisely measure the new diameter and record it for  $D$ . Repeat this several times. Make sure that you record the *cumulative* length; that is, if you unroll about 3 meters each time, the values for  $L$  should be about 3, 6, 9, and so on. Stop when the roll is about half gone.
3. Measure the diameter of the core—the spool that the paper is wound on. Record this measurement below the table. You’ll need it later.

	D	L
units	millimeters	meters
1		0 m
2		
3		
4		
5		
6		
7		
8		
9		

diameter of core = \_\_\_\_\_ mm

## INVESTIGATE

Next you'll analyze the data with Fathom.

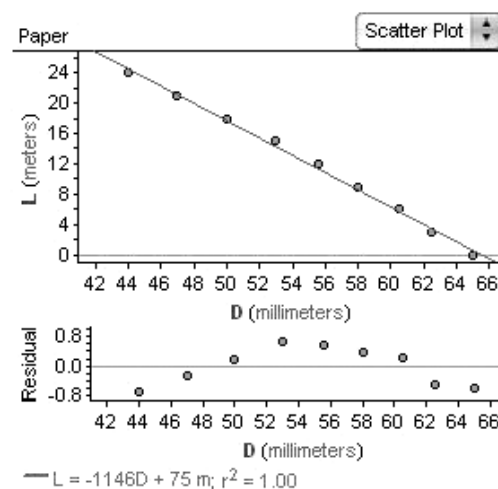
4. Start Fathom and enter your data into a case table. Use the attributes  $D$  and  $L$ , and use units, as shown in the table.
5. Make a scatter plot of  $L$  versus  $D$ . The data probably look fairly linear.
6. With the graph selected, choose **Least-Squares Line** from the **Graph** menu.

You can use a residual plot to help you judge how well the function fits the data.

7. Choose **Make Residual Plot** from the **Graph** menu.

- Q2 Describe what you see in the residual plot. Based on the residual plot, do you think the least-squares line is a good fit?
- Q3 Compare what you have in your graph so far to your prediction in question Q1. Which aspects of the Fathom graph meet your expectations? Which do not?

Here's a sample graph from step 7. Your graph probably looks similar. Notice that even though the least-squares line has an  $r^2$  close to 1.00, there is something wrong with the line as a model. The residuals form an obvious and consistent bow, which you can barely see in the top graph. When the residuals have an obvious pattern like this, it usually means that the function model is not correct.



To find a better model, think about what aspects of the circular roll change as you remove paper. The diameter of the circle changes, but the area also changes—pulling off paper removes area from the outside of the circle. Area is a square dimension, so see whether squaring the variables helps make a better fit.

8. Make two new attributes,  $DD$  and  $LL$ , that are defined by the formulas  $D^2$  and  $L^2$ , respectively.
9. Make scatter plots of all combinations of  $L$  or  $LL$  (vertical) versus  $D$  or  $DD$  (horizontal). Add a least-squares line and residual plot to each scatter plot.
- Q4 Sketch the four possible graphs. Which graph is closest to a straight line?

## How Much Paper Is Left?

(continued)

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- Q5** Write the equation of the least-squares line for the graph in Q4 that was most linear.
- Q6** When you're out of paper,  $D$  will be the diameter of the core. When you're out of paper, what value do you need to substitute into the equation in Q5? (*Hint: Pay attention to units!*)
- Q7** Use your answers to Q5 and Q6 to substitute and calculate how much paper is on the roll.
- Q8** Unroll the entire roll and measure the total length of paper. Or, if your teacher has the packaging from the adding-machine rolls, find out how much paper is supposed to be on each roll. How does your prediction in Q7 compare to the actual length of paper on the roll?
- Q9** In step 5, the scatter plot of  $L$  versus  $D$  looked fairly linear even though the relationship is *not* linear. How can it be that the data looked so straight? If you had collected more data, do you think you would have recognized the curve more clearly? Explain why or why not.

## EXPLORE MORE

1. Most adding-machine rolls have a band of red on the paper nearest the core. When you use the roll in an adding machine, calculator, or cash register, you know you are about to run out of paper when you see a red edge on the paper. Calculate how much paper is left on the roll when you first see the red edge.
2. The equation that you wrote in Q5 is the least-squares line for *transformed* data. Use algebra to turn the linear function into a curved function for the original data. What type of function do you get? Does that function make sense based on the geometry of this situation? Plot the curved function on your original graph of  $L$  versus  $D$ . How well does the curved function fit your data?

# Compound Interest

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*Interest* is a charge that you pay a bank for borrowing money, or that the bank pays you for keeping money in your bank account and letting the bank invest and earn interest on it. Interest rates are usually stated as annual, or yearly, rates. *Simple interest* is calculated by multiplying the initial investment (the *principal*) times the interest rate times the length of time. With simple interest, the interest earned does not become part of your principal. The interest is always calculated as a percentage of the original principal.

Most banks, however, use *compound interest*. This means the interest earned is added to your principal, so that interest is calculated on a growing amount of money. You probably already know that compound interest is an exponential function. For example, if you invest \$2000 at an annual interest rate of 7%, your *balance* over the years is

Year 0: \$2000

Year 1:  $\$2000(1 + 0.07)$

Year 2:  $[\$2000(1 + 0.07)](1 + 0.07) = \$2000(1 + 0.07)^2$

Year 3:  $[\$2000(1 + 0.07)^2](1 + 0.07) = \$2000(1 + 0.07)^3$

Year  $n$ :  $\$2000(1 + 0.07)^n$

Many banks calculate interest on an account or a loan more frequently than once a year. In these cases, the annual interest rate is divided quarterly (by 4), monthly (by 12), or by another number that corresponds to the number of periods per year. This creates an exponential function, too. For example, if the annual rate of 7% is compounded monthly, then your balance over the months is

Month 0: \$2000

Month 1:  $\$2000\left(1 + \frac{0.07}{12}\right)$

Month 2:  $\$2000\left(1 + \frac{0.07}{12}\right)^2$

Month 3:  $\$2000\left(1 + \frac{0.07}{12}\right)^3$

Month  $n$ :  $\$2000\left(1 + \frac{0.07}{12}\right)^n$

When you add money to the account—or make *payments*—your interest is calculated from the previous month’s balance plus the payment. Here’s an example with a monthly payment of \$500:

Month 0: \$2000

Month 1:  $\$2000\left(1 + \frac{0.07}{12}\right) + \$500$

Month 2:  $\left[\$2000\left(1 + \frac{0.07}{12}\right) + \$500\right]\left(1 + \frac{0.07}{12}\right) + \$500$

Month 3:  $\left[\left[\$2000\left(1 + \frac{0.07}{12}\right) + \$500\right]\left(1 + \frac{0.07}{12}\right) + \$500\right]\left(1 + \frac{0.07}{12}\right) + \$500$

Month  $n$ :  $\$2000\left(1 + \frac{0.07}{12}\right)^n + \$500\left(1 + \frac{0.07}{12}\right)^{n-1} + \$500\left(1 + \frac{0.07}{12}\right)^{n-2} + \cdots + \$500$

In this activity, you’ll use Fathom to explore compound interest. By watching the balance change as you adjust the parameters, you’ll get a better understanding of compound interest.

## INVESTIGATE

1. Open the document **CompoundInterest.ftm**.

The sliders in this document change the principal (*Principal*), the annual interest rate (*InterestRate*), the compounding periods (*PeriodsPerYear*), and the payment (*PaymentPerPeriod*). The case table uses formulas to show time in different formats, the current balance, the interest that *accrues* (is earned) during each period, and the payment that is made during each period. The graph shows *Balance* as a function of *TimeInYears*.

2. Play with the sliders, individually and in combination with each other. Get a feel for how each slider affects the values in the case table and the graph. If you use values that make points go off the graph, choose **Rescale Graph Axes** from the **Graph** menu to see the entire graph.

**Q1** How does *Principal* affect the graph of *Balance* versus *TimeInYears*?

**Q2** How does *InterestRate* affect the graph? What happens if you use a negative value of *InterestRate*? What would a negative interest rate mean in a real-world situation?

**Q3** How does *PeriodsPerYear* affect the graph?

- Q4** How does *PaymentPerPeriod* affect the graph? What happens if you use a negative value of *PaymentPerPeriod*? What would a negative payment mean in a real-world situation?

Now that you know how the sliders work, try using them to solve some specific problems.

- Q5** Adjust the sliders to solve one or more of these problems. Along with your answer, briefly explain how you set the sliders and found the solution.
- a.** Suppose \$3000 is invested in a retirement account that earns 2.5% annual interest, compounded quarterly. You make no additional contributions to the account. What is the balance after the 3rd quarter of the 5th year?
  - b.** You enroll in an investment plan that deducts \$100 from your monthly paycheck and deposits it into an account that earns 7%, compounded monthly. How much will you have invested after your 25th paycheck?
  - c.** You borrow \$1200 at 4.5% annual interest, compounded monthly. If you must pay off the loan within 3 years, how much will you need to pay each month? (*Hint:* For a loan, your payments *reduce* the balance.)
  - d.** Kelly buys \$2000 of furniture on her credit card. The card has an annual interest rate of 18%, compounded monthly. If Kelly makes only the minimum payment of \$40 each month, when will she pay off the balance? How much will she have actually paid for the furniture, including all of the interest? (*Hint:* In order to find the answer, you'll need to add several months to your case table. Select the case table and choose **New Cases** from the **Collection** menu to do this.)

## EXPLORE MORE

- 1.** For a given principal and interest rate, how much more is the balance at the end of 3 years compounding once per year versus quarterly? How does monthly compare? How about weekly (52 times per year) and daily (usually taken as 360 times per year)? Complete this sentence: In the long run, if your interest is compounded more frequently, your balance will be . . .

## Compound Interest

(continued).

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2. Investigate the relationship between interest rate and *doubling time*—the time it takes for your balance to be twice as much as your principal. First explore interest compounded yearly, then quarterly, and then monthly. Regardless of your principal, can you figure out doubling time based on the compound interest rate? Is it possible to figure out what compound interest rate is required to have the balance double after a certain amount of time, say, 3 years? (*Note:* Keep *PaymentPerYear* set to 0 for this exploration.)
3. Investigate the relationship between payment per period and the balance after 3 years. Be as quantitative as possible. Explore interest compounded yearly, then quarterly, and then monthly.